Part 1 · Integration and the Riesz Representation Theorem

1. 0. Introduction to Part 1

A hint of the flavour of abstract harmonic analysis can (as is indicated in Edwards [4]) be transmitted as soon as a relatively primitive concept of invariant integration on groups is available; for the hint to get across, it suffices that one can integrate (say) continuous functions with compact supports. In order to make a more serious study, it is necessary (as was indicated loc. cit.) to have a more highly developed integration theory of the Lebesgue type. The major aim of Part 1 is to provide a brief account of one way of extending a primitive integration theory into such a Lebesgue-type theory.

This aim might be attained in any one of several ways. The chosen method might be said to be that which fulfils most expeditiously the secondary aim of exhibiting some aspects of the general role of integration theory in functional analysis and abstract analysis in general. Since this role is to a large extent crystallised in the so-called Riesz representation theorem (RRT, for short), the selected approach to integration theory is accordingly the one which is dominated by the idea of viewing integration as a linear functional defined on a space of continuous functions. This approach is in opposition to accounts which (cf. 1.1 below) base integration on a given measure function: instead, the measure function is made to appear as a derivative concept.

No attempt will be made to present this approach in the most general setting possible; in fact, we shall assume (except in various 'asides') that the underlying space is compact and Hausdorff. This restriction brings with it a number of technical simplifications, while yet preserving enough generality to bring out most of the important features and to ensure general utility. (Extensions and historical remarks will appear in 1.9 and 1.10, respectively.)

To come closer to particularities, the representation problem we intend to tackle is the following. Suppose X is a set. Denote by $B(X)$ the linear space of bounded real-valued functions on X. Let L denote a linear subspace of $B(X)$. Consider linear functionals F on L which are continuous in the sense that

$$|F(f)| \leq \text{const.}\ \|f\| = \text{const.}\ \sup_{x \in X} |f(x)|$$

for every $f \in L$. If L is finite dimensional, this continuity requirement is fulfilled by every linear functional F on L, and F is expressible as a weighted sum:

$$F(f) = \sum_{j=1}^{n} c_j f(x_j)$$

for suitably chosen $x_j \in X$ and real numbers c_j. If L is not finite dimensional, it is too much to expect that such a representation is always possible. However, one might hope that every continuous linear F will be expressible as some sort of integral.

In 1.1 we shall show quite rapidly and painlessly that this hope is justified in the shape of the Hildebrandt-Fichtenholz-Kantorovich theorem, at least for suitable choices of L. (The required concept of integration will be defined on the way.) However, for reasons which will be pinpointed in 1.1.10, this solution is not as helpful as one might wish.

It turns out that, in certain special and very important cases, it is possible to adopt a more painstaking and more constructive approach leading to a result (the Riesz representation theorem) of the desired type which is free from shortcomings of the earlier solution. The remainder of Part 1 is concerned with this more profitable approach.

1.1. Preliminaries regarding measures and integrals

Throughout 1.1, X denotes an arbitrarily given nonvoid set. The term 'set', without further qualification, means 'subset of X'. $\mathscr{P}(X)$ is a convenient symbol for denoting the set of all subsets of X. Throughout 1.1-1.10 only real-valued functions will be considered. The extension to complex-valued functions is a routine matter; see 1.11 below.

1.1.1. By an algebra of sets is meant a set $\mathscr{A}$ of subsets of X containing X itself as a member and stable under (finite) unions and under complementation. If in addition $\mathscr{A}$ contains as a member the union of any denumerable sequence of its members, $\mathscr{A}$ is termed a σ-algebra.

A set-function on $\mathscr{A}$ means simply a real-valued function whose domain is $\mathscr{A}$. (Complex-valued set-functions appear later; see 1.11.) Such a set-function μ is said to be (finitely) additive if $\mathscr{A}$ is an algebra and if

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever A and B are disjoint members of $\mathscr{A}$. If moreover

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

whenever (A_n) is a disjoint sequence of members of $\mathscr{A}$ whose union belongs to $\mathscr{A}$, then μ is said to be σ-additive (= countably additive, or completely additive) on $\mathscr{A}$.

A set-function μ on $\mathscr{A}$ is said to have bounded variation (BV for short) on $\mathscr{A}$ if

$$V(\mu) \equiv \sup \sum_{k=1}^{n} |\mu(A_k)|$$

is finite, the supremum being taken with respect to all finite disjoint families $(A_k)_{1 \le k \le n}$ of members of $\mathscr{A}$. This is certainly the case if μ is non-negative and additive. (Non-negative set-functions are sometimes allowed to take the value ∞, but if they do they are no longer of BV. They are barred from further discussion here.)

For brevity an additive (resp. σ-additive) set-function of BV on an algebra (resp. σ-algebra) $\mathscr{A}$ will be termed a measure (resp. σ-measure) on $\mathscr{A}$.

1.1.2. **Examples.** Since it turns out that the great virtue of the Riesz representation theorem, when compared with the results of 1.1.9 below, hinges upon the difference between additivity and σ-additivity (see 2.1.1 below), some examples exhibiting the difference are in order.

(i) If X is a finite set, it is evident that any additive set-function on any algebra $\mathscr{A}$ of subsets of X is also σ-additive. This is false for every infinite set X, however. Counter-examples can be produced by using transfinite methods, such as the Hahn-Banach theorem (if I is as in the proof of 1.9.8 below, and if c_A denotes the characteristic function of A, then $\mu: A \mapsto I(c_A)$ is additive and not σ-additive) or the use of ultrafilters (see Edwards [2], Exercises 1.23 and 1.26).

All examples of this sort may seem somewhat artificial. The next one is somewhat more natural.

(ii) For X take the real interval $[0, 1]$. Let $\mathscr{A}$ consist of all subsets of X which are finite unions of subintervals of X. (The intervals may be void and may contain neither, either one, or both extremities.) $\mathscr{A}$ is an algebra of subsets of X, but not a σ-algebra.

Let g be any real-valued function on X. If J is a subinterval of X with left and right extremities a and b respectively, write $\Delta g(J) = g(b) - g(a)$. Each member A of $\mathscr{A}$ can be written as a finite union of disjoint subintervals of X, say $A = J_1 \cup \dots \cup J_n$. The sum $\sum_{k=1}^{n} \Delta g(J_k)$ can be shown to be independent of the chosen decomposition of A. A set-function μ is accordingly unambiguously defined on $\mathscr{A}$ by putting $\mu(A) = \sum_{k=1}^{n} \Delta g(J_k)$. It can be verified that μ, so defined, is additive on $\mathscr{A}$.

However, if g be suitably chosen, μ will not be σ-additive. Thus, suppose either that $\lim g(\frac{1}{n})$ does not exist, or that it exists and is different from $g(0)$. The subinterval $J = (0, 1]$ is the union of the disjoint subintervals $J_n = (\frac{1}{n+1}, \frac{1}{n}]$ $(n = 1, 2, \dots)$. Were μ to be σ-additive, the relation

$$\mu(J) = \sum_{n=1}^{\infty} \mu(J_n)$$

would follow. Now $\mu(J) = g(1) - g(0)$, $\mu(J_n) = g(\frac{1}{n}) - g(\frac{1}{n+1})$, so that the series $\sum_{n=1}^{\infty} \mu(J_n)$ either does not converge or, if it does, converges to the sum $g(1) - \lim g(\frac{1}{n})$, which is different from $g(1) - g(0)$. Thus μ is not σ-additive.

Notice that μ will have BV if and only if g has bounded variation on $[0, 1]$.

That $\mathscr{A}$ itself is not a σ-algebra is of no ultimate significance, for the Hahn-Banach theorem could be used to show that μ can be extended into a positive, additive set-function on the σ-algebra $\mathscr{P}(X)$.

1.1.3. **The space** $B_{\mathscr{A}}(X)$. Let $\mathscr{A}$ be an algebra of sets. By an $\mathscr{A}$-function (or $\mathscr{A}$-simple function) will be meant a function on X which is representable as a finite linear combination (with real coefficients) of characteristic functions c_A, where $A \in \mathscr{A}$. Herein c_A is that function on X which takes the value 1 at points belonging to A and the value zero at all remaining points of X.

The set $B(X)$ of all bounded real-valued functions on X is a Banach space, addition of functions and multiplication of a function by a real scalar being defined as usual ('pointwise operations'), and the norm being defined by

$$\|f\| = \sup_{x \in X} |f(x)| . \qquad (1.1.1)$$

It is evident that $B(X)$ contains each $\mathscr{A}$-function.

$B_{\mathscr{A}}(X)$ will denote the closure in $B(X)$ of the set of all $\mathscr{A}$-functions.

It can be shown that if $\mathscr{A}$ is a σ-algebra, then $B_{\mathscr{A}}(X)$ contains precisely those $f \in B(X)$ which are $\mathscr{A}$-measurable in the sense that

$$\{x \in X : f(x) > r\} \in \mathscr{A}$$

for each real number r. In particular, if $\mathscr{A} = \mathscr{P}(X)$, $B_{\mathscr{A}}(X) = B(X)$.

1.1.5. **Exercise.** Prove the last two statements.

The next step is to consider the definition of the integral $\int f d\mu$ for $f \in B_{\mathscr{A}}(X)$, μ being any measure on the algebra $\mathscr{A}$.

1.1.6. **Definition of** $\int f d\mu$. Take first the case in which f is an $\mathscr{A}$-function. It then admits at least one (and actually many) expressions as a finite sum

$$f = \Sigma_{k=1}^{n} \alpha_k \cdot c_{A_k} ,$$

the α_k being real numbers and the A_k members of $\mathscr{A}$. The additivity of μ on $\mathscr{A}$ is easily seen to ensure that the associated sums

$$\Sigma_{k=1}^{n} \alpha_k \mu(A_k)$$

are independent of the selected expression of f. The common value of these sums is, by definition, the meaning of the symbol $\int f d\mu$.

With this convention it is evident that the functional

$$f \mapsto \int f d\mu$$

is linear on the vector space of $\mathscr{A}$-functions, and that

$$\left| \int f d\mu \right| \leq V(\mu) \, . \, \|f\| \qquad (1.1.2)$$

for all $\mathscr{A}$-functions f.

Take next any $f \in B_{\mathscr{A}}(X)$. Choose any sequence (f_n) of $\mathscr{A}$-functions converging uniformly to f, i. e., such that $\lim \|f-f_n\| = 0$. Such sequences do exist by virtue of the definition of $B_{\mathscr{A}}(X)$. From (1.1.2) it follows that the sequence $(\int f_n d\mu)$ is convergent (to a finite limit), and moreover that this limit is independent of the chosen sequence (f_n) converging uniformly to f. Accordingly, $\int f d\mu$ may and will be unambiguously defined to be this common limit.

1.1.7. Exercise. Verify in detail the statements made in the penultimate sentence. Check also that (1.1.2) continues to hold for any $f \in B_{\mathscr{A}}(X)$.

1.1.8. Exercise. Show that $f \mapsto \int f d\mu$ is a continuous linear functional on $B_{\mathscr{A}}(X)$, the latter regarded as a Banach space with the norm induced on it by (1.1.1).

1.1.9. The space dual to $B_{\mathscr{A}}(X)$. The result stated in Exercise 1.1.8 has a valid converse, namely: any continuous linear functional F on $B_{\mathscr{A}}(X)$ is expressible by integration with respect to some measure μ on $\mathscr{A}$.

Indeed, if this is to be the case, only one choice of the set-function μ is possible, namely

$$\mu(A) = F(c_A) . \qquad (1.1.3)$$

It remains to be shown that μ, so defined, is a measure on $\mathscr{A}$ and that

$$F(f) = \int f d\mu \qquad (1.1.4)$$

for $f \in B_{\mathscr{A}}(X)$.

It is evident from (1.1.3) and the linearity of F, that μ is additive on $\mathscr{A}$. To prove it has BV, suppose that $(A_k)_{1 \le k \le n}$ is a finite disjoint family of members of $\mathscr{A}$. Define the numbers $\alpha_k = \operatorname{sgn} \mu(A_k)$. Then

$$\| \Sigma_{k=1}^{n} \alpha_k c_{A_k} \| \le 1$$

and so

$$\Sigma_{k=1}^{n} |\mu(A_k)| = F(\Sigma_{k=1}^{n} \alpha_k c_{A_k}) \le \| F \| ,$$

by linearity of F and the standard definition

$$\| F \| = \sup \{ |F(f)| : f \quad B_{\mathscr{A}}(X), \quad \| f \| \le 1 \} .$$

It appears thence that $V(\mu) \le \| F \|$, so that μ has BV. μ is therefore a measure on $\mathscr{A}$.

Now (1.1.3), combined with the linearity of F and the linearity of the integration process, shows that (1.1.4) holds when f is any $\mathscr{A}$-function. Then, by continuity of F and by continuity of the integration process (Exercise 1.1.8), (1.1.4) must continue to hold for any $f \in B_{\mathscr{A}}(X)$.

This establishes the opening statement of this section.

A simple argument shows further that

$$\| F \| = V(\mu) .$$

To sum up, one may say that formula (1.1.4) establishes a linear isometry $F \leftrightarrow \mu$ between the dual (= conjugate, or adjoint) space of $B_{\mathscr{A}}(X)$ and the space of all measures on $\mathscr{A}$, the latter space of measures carrying the norm defined by

$$\|\mu\| = V(\mu) .$$

This result is sometimes referred to as the Hildebrandt-Fichtenholz-Kantorovich theorem.

1.1.10. Some notation. We shall agree on the following notation for subsequent use.

If X is any set and k any 'object', the constant function with domain X and range $\{k\}$ will be denoted by $\underline{k}_X$, or simply by $\underline{k}$ if X is clear from the context. As a set of ordered pairs, $\underline{k}_X$ is thus $X \times \{k\}$.

If f and g are real-valued functions on X, we write $f \leq g$ (or $g \geq f$) if and only if $f(x) \leq g(x)$ for every $x \in X$. Thus, if X is a compact space, a linear functional F on $C(X)$ is non-negative (as defined immediately following 1.2.2 below) if and only if $F(f) \geq 0$ for every $f \in C(X)$ satisfying $f \geq \underline{0}_X$.

1.1.11. Exercise. Let μ be a non-negative measure on $\mathscr{P}(X)$ which is not σ-additive (see 1.1.2). Exhibit a monotone sequence (f_j) extracted from $B(X)$ such that $\underline{0} \leq f_j \leq \underline{1}$, $f = \lim_{j\to\infty} f_j = \underline{0}$ (the constant function zero),

$$\inf_{j \in N} \int f_j d\mu > 0 ,$$

and therefore

$$\lim_{j\to\infty} \int f_j d\mu \neq \int f d\mu$$

Remark. This example shows that passage to the limit under the integral sign is not generally permissible with integrals with respect to finitely additive measures, even though the integrands are quite well behaved and form a monotone sequence. This disagreeable situation is much improved when integrals with respect to σ-additive measures are considered; cf. Theorem 1.5.3 below.

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Excerpt
More information

1. 2. Statement and discussion of Riesz's theorem.

1. 2. 1. Statement of the problem. A functional analytic approach to problems related to classical analysis often focuses attention on the situation in which X is a fairly simple type of topological space (rather than a structureless set) and B(X) is replaced by its subspace C(X) comprising all bounded continuous real-valued functions on X. (Notice that B(X) = C(X), if the set X is endowed with its discrete topology.) It is often of importance, and is in any case of considerable intrinsic interest, to know what the continuous linear functionals on C(X) look like.

One answer to this problem flows immediately from the substance of 1. 1. 9, if one applies the Hahn-Banach theorem. If F is a CLF (= continuous linear functional) on C(X), it has an extension into a CLF on the whole of B(X). So, by 1. 1. 9, there exists a measure μ on $\mathscr{P}$(X) such that (1. 1. 4) holds for all f $\in$ C(X). See also Edwards [4], §1.

This reply is unsatisfactory from several points of view. To begin with, μ is very far from being uniquely determined by F, i. e., there will exist in general many measures μ which represent, via (1. 1. 4), that CLF F which is identically zero on C(X). (If X is normal, this defect can be removed by restricting μ to be 'regular', as defined below.)

A second and more important criticism stems from the preference for a representation which will be useful. As Exercise 1. 1. 11 indicates, integrals $\int \dots d\mu$ with respect to arbitrary finitely-additive measures μ can behave very oddly, most of the nice theorems (like 1. 5. 3, 1. 5. 4 and 1. 8. 1 below) for Lebesgue-type integrals suffering spectacular breakdowns. For this reason it is natural to consider the possibility of a representation (1. 1. 4) in which μ is a σ-measure on some σ-algebra, $\mathscr{A}$, such that $C(X) \subseteq B_{\mathscr{A}}(X)$. This inclusion relation will obtain if (and, for the simpler examples of X, only if) $\mathscr{A}$ contains all the so-called Borel subsets of X; see 1. 9. The Borel subsets of X are, by definition, precisely the elements of the smallest σ-algebra of subsets of X which contains all open (or all closed) subsets of X.

The suggestion is therefore that we define the term Borel measure (on X) to mean a σ-measure with domain the set of all Borel sets in X, and then ask whether formula (1. 1. 4) holds with μ a suitably chosen Borel measure. [Note - The term 'Borel measure' is often applied, even when μ is not of BV, but this extended meaning will not be used here.]

Granted such a representation, the uniqueness of μ for a given F would not be ensured unless one imposed the extra condition that μ be regular in the sense that, for any Borel set $A \subseteq X$ and any $\varepsilon > 0$, there exists an open set $U \supseteq A$ such that $|\mu(B) - \mu(A)| \leq \varepsilon$ for every Borel set B satisfying $A \subseteq B \subseteq U$.

The statement of what has come to be known briefly as the Riesz representation theorem (hereinafter referred to even more briefly as the RRT) is just a summary of the desiderata outlined above, together with hypotheses on X which suffice to render the goal attainable.

1.2.2. **Theorem.** Let X be a Hausdorff compact space and F any CLF on C(X). Then there exists a regular Borel measure μ on X such that (1.1.4) holds for $f \in C(X)$. Moreover, μ is uniquely determined by F, and $V(\mu) = \|F\|$.

If the existence of μ be assumed, it is not difficult to show that μ is non-negative whenever F is non-negative in the sense that $F(f) \geq 0$ for every $f \in C(X)$ satisfying $f \geq 0_X$. (Incidentally, many writers use the term 'positive' in place of 'non-negative' in this connection. For obvious reasons, either choice is regrettable, the second slightly less so than the first. A better term would be '(monotone) nondecreasing', but this is non-standard.) The converse is trivial. Since it may be shown (see Exercise 1.2.6 (b) below) that any CLF F on C(X) is expressible as the difference of two (necessarily continuous) non-negative linear functionals on C(X), an equivalent formulation of the existence part of the RRT appears in

1.2.3. **Theorem.** If X is any Hausdorff compact space, any non-negative linear functional F on C(X) is representable in the form (1.1.4), where μ is a non-negative regular Borel measure on X. (As before, μ is uniquely determined by F, etc.)

In Theorem 1.2.3 it is unnecessary to postulate continuity of F since, as is easily seen, this is a consequence of non-negativity; see Exercise 1.2.6 (b).

The ensuing programme is confined to providing a proof of the