LECTURE 1

Introduction. Plan of the lectures. Poisson structures

The theory of Solitons ("solitary waves") deals with the propagation of non-linear waves in continuum media. Their famous discovery has been done in the period 1965-1968 (by M. Kruscal and N. Zabuski, 1965; G. Gardner, I. Green, M. Kruscal and R. Miura, 1967; P. Lax, 1968 —see the survey [8] or the book [22]).

The familiar KdV (Korteweg-de Vries) non-linear equation was found to be exactly solvable in some profound nontrivial sense by the so-called "Inverse Scattering Transform" (IST) at least for the class of rapidly decreasing initial data $\Phi(x)$:

$$\Phi_t = 6\Phi\Phi_x - \Phi_{xxx} \qquad \text{[KdV]}.$$

Some other famous systems are also solvable by analogous procedures. The following are examples of $(1+1)$-systems:

$$\begin{aligned} &\Phi_{\eta\varsigma} = \sin\Phi && \text{[SG]} \\ &i\Phi_t = -\Phi_{xx} \pm |\Phi|^2\Phi && [\text{NS}_\pm] \\ &\dots . \end{aligned}$$

The most interesting integrable $(2+1)$-systems are the following:

$$\begin{cases} w_x = u_y \\ u_t = 6uu_x + u_{xxx} + 3\alpha^2 w_y \\ \alpha^2 = \pm 1 \end{cases} \qquad \text{[KP]},$$

$$\begin{cases} V_t = \Re(V_{zzz} + (aV)_z) \\ \overline{V} = V, a_{\overline{z}} = 3V_z, \partial_z = \partial_x + i\partial_y \end{cases} \qquad \text{[2-dimKdV]}.$$

There are different beautiful connections between Solitons and Geometry, which we will now shortly describe.

Solitons and 3-dimensional Geometry.

a) The sine-Gordon equation $\Phi_{\eta\varsigma} = \sin \Phi$ appeared for the first time in a problem of 3-dimensional geometry: it describes locally the isometric imbeddings of the Lobatchevski 2-plane L^2 (i.e. the surface with constant negative Gaussian curvature) in the Euclidean 3-space $\mathbb{R}^3$. Here Φ is the angle between two asymptotic directions (η, ς) on the surface along which the second (curvature) form is zero. It has been used by Bianchi, Lie and Backlund for the construction of new imbeddings ("Backlund transformations", discovered by Bianchi).

b) The elliptic equation

$$\Delta\Phi = \sinh \Phi$$

appeared recently for the description of genus 1 surfaces ("topological tori") in $\mathbb{R}^3$ with constant mean curvature (H. Wente, 1986; R. Walter, 1987).

Starting from 1989 F. Hitchin, U. Pinkall, N. Ercolani, H. Knorrer, E. Trubowitz and A. Bobenko used in this field the technique of the "periodic IST" —see [5].

Solitons and algebraic geometry.

a) There is a famous connection of Soliton theory with algebraic geometry. It appeared in 1974-1975. The solution of the periodic problems of Soliton theory led to beautiful analytical constructions involving Riemann surfaces and their Jacobian varieties, Θ-functions and later also Prym varieties and so on (S. Novikov, 1974; B. Dubrovin-S. Novikov, 1974; B. Dubrovin, 1975; A. Its-V. Matveev,1975; P. Lax, 1975; H. McKean-P. Van Moerbeke, 1975 —see [8]).

Many people worked in this area later (see [22], [8], [7] and [6]). Very important results were obtained in different areas, including classical problems in the theory of Θ-functions and construction of the harmonic analysis on Riemann surfaces in connection with the "string theory" —see [15], [16], [17] and [6].

b) Some very new and deep connection of the KdV theory with the topology of the moduli spaces of Riemann surfaces appeared recently in the works of M. Kontzevich (1992) in the development of the so-called "2-d quantum gravity". It is a byproduct of the theory of "matrix models" of D. Gross-A. Migdal, E. Bresin-V. Kasakov and M. Duglas - N. Shenker, in which Soliton theory appeared as a theory of the so-called "renormalization group" in 1989-90.

Soliton theory and Riemannian geometry.

Let us recall that the systems of Soliton theory (like KdV) sometimes describe the propagation of non-linear waves. For the solution of some problem we are going to develop an asymptotic method which may be considered as a natural non-linear analogue of the famous WKB approximation in Quantum Mechanics. It leads to the structures of Riemannian Geometry; some nice classes of infinite-dimensional Lie algebras appeared in this theory. This will be exactly the subject of the present lectures (see also [10]). *All beautiful constructions of Soliton theory are available for Hamiltonian systems only (nobody knows why).* Therefore we will start with an elementary introduction to Symplectic and Poisson Geometry (see also [21], [7], [10] and [11]).

Plan of the lectures

- Symplectic and Poisson structures on finite-dimensional manifolds. Dirac monopole in classical mechanics. Complete integrability and Algebraic Geometry. This is the subject of Lectures 1, 2 and 3.
- Local Poisson Structures on loop spaces. First-order structures and finite-dimensional Riemannian Geometry. Hydrodynamic-Type systems. Infinite-dimensional Lie Algebras. Riemann Invariants and classical problems of Differential Geometry. Orthogonal coordinates in $\mathbb{R}^n$. (Lecture 4.)
- Nonlinear analogue of the WKB-method. Hydrodynamics of Soliton Lattices. Special analysis for the KdV equation. Dispersive analogue of the shock wave. Genus 1 solution for the hydrodynamics of Soliton Lattices. (Lecture 5.)

Symplectic and Poisson structures

Let M be a finite-dimensional manifold with a system $(y^1, \ldots, y^m)$ of (local) coordinates.

DEFINITION 1.1. Any non-degenerate closed 2-form

$$\Omega = \omega_{\alpha\beta} dy^\alpha \wedge dy^\beta$$

generates a *symplectic structure* on the manifold M. Non-degeneracy means exactly that the skew-symmetric matrix $(\omega_{\alpha\beta})$ is non-singular for all points $y \in M$, i.e.

$$\det(\omega_{\alpha\beta}(y)) \neq 0.$$

REMARK 1.2. Since a skew-symmetric matrix in odd dimension is necessarily singular we have that if M has a symplectic structure then it has even dimension.

By definition a symplectic structure is just a special skew-symmetric scalar product of the tangent vectors: if $V = (V^\alpha)$ and $W = (W^\beta)$ are coordinates of tangent vectors we set:

$$(V, W) = \omega_{\alpha\beta} V^\alpha W^\beta = -(W, V).$$

Let $\omega^{\alpha\beta}$ denote the inverse matrix:

$$\omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^{\alpha}_{\beta}.$$

This inverse matrix $(\omega^{\alpha\beta})$ determines everything important in the theory of Hamiltonian systems. Therefore we shall start with the following definition.

DEFINITION 1.3. A skew-symmetric C^∞-tensor field $(\omega^{\alpha\beta})$ on the manifold M generates a *Poisson structure* if the Poisson bracket (defined below) turns the space $C^\infty(M)$ into a Lie algebra: for any two functions $f,\ g \in C^\infty(M)$ we define their *Poisson bracket* as a scalar product of the gradients:

$$\{f,g\} = \omega^{\alpha\beta}\frac{\partial f}{\partial y^\alpha}\frac{\partial g}{\partial y^\beta} = -\{g,f\}.$$

This operation obviously satisfies the following requirements:

$$\{f,g\} = -\{g,f\},$$
$$\{f+g,h\} = \{f,h\}+\{g,h\},$$
$$\{fg,h\} = f\{g,h\}+g\{f,h\},$$

so only the Jacobi identity is non-obvious.

REMARK 1.4. Using the coordinate functions we have that

$$\{y^\alpha, y^\beta\} = \omega^{\alpha\beta}$$
$$\{\{y^\alpha, y^\beta\}, y^\gamma\} = \frac{\partial \omega^{\alpha\beta}}{\partial y^k}\frac{\partial y^\gamma}{\partial y^p}\omega^{kp} = \frac{\partial \omega^{\alpha\beta}}{\partial y^k}\omega^{k\gamma}$$

and it is easily checked that the Jacobi identity is equivalent to:

$$\frac{\partial \omega^{\alpha\beta}}{\partial y^k}\omega^{k\gamma} + \frac{\partial \omega^{\gamma\alpha}}{\partial y^k}\omega^{k\beta} + \frac{\partial \omega^{\beta\gamma}}{\partial y^k}\omega^{k\alpha} = 0 \quad \forall \alpha,\beta,\gamma.$$

It is an elementary algebraic exercise to check the following: In case $(\omega^{\alpha\beta})$ is non-singular and $(\omega_{\alpha\beta})$ denotes the inverse matrix the Jacobi identity is also equivalent to:

$$\frac{\partial \omega_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial \omega_{\gamma\alpha}}{\partial y^\beta} + \frac{\partial \omega_{\beta\gamma}}{\partial y^\alpha} = 0 \quad \forall \alpha,\beta,\gamma$$

i.e. to

$$d\left(\sum_{\alpha<\beta}\omega_{\alpha\beta}dy^\alpha \wedge dy^\beta\right) = 0$$

i.e. to closedness of the 2-form $\omega_{\alpha\beta} dy^\alpha \wedge dy^\beta$. (Recall however that the inverse matrix does not exist in some important cases.)

DEFINITION 1.5. A function $f \in C^\infty(M)$ is called a *Casimir* for the given Poisson bracket if it belongs to the kernel (or annihilator) of the Poisson bracket, i.e. if for any function $g \in C^\infty(M)$ we have

$$\{f, g\} = 0.$$

LECTURE 2

Poisson Structures on Finite-dimensional Manifolds. Hamiltonian Systems. Completely Integrable Systems

As in Lecture 1 we are dealing with a finite-dimensional manifold M with (local) coordinates $(y^1, \dots, y^m)$ and a Poisson tensor field $-\omega^{ij} = \omega^{ji}$ such that the corresponding Poisson bracket

$$\{f, g\} = \omega^{ij} \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j}$$

generates a Lie algebra structure on the space $C^\infty(M)$.

DEFINITION 2.1. A smooth function $H(y)$ on M (or, more in general, a closed 1-form $H_\alpha dy^\alpha$, where $H_\alpha = \partial H/\partial y^\alpha$ if H exists) generates a *Hamiltonian system* by the formula

$$\dot{y}^\alpha = \omega^{\alpha\beta} H_\beta.$$

For any function $f \in C^\infty(M)$ we define

$$\dot{f} = \{f, H\} = \omega^{\alpha\beta} H_\beta f_\alpha.$$

DEFINITION 2.2. We will say a vector field V with coordinates (V^α) is a *Hamiltonian vector field generated by the Hamiltonian* $H \in C^\infty(M)$ if

$$V^\alpha = w^{\beta\alpha} \partial H/\partial y^\beta.$$

A well-known lemma states that the commutator of any pair of Hamiltonian vector fields is also Hamiltonian and it is generated by the Poisson bracket of the corresponding Hamiltonians. (The reader may verify this lemma by direct calculation.)

DEFINITION 2.3. We define a *Poisson algebra* as a commutative associative algebra C with an additional Lie algebra operation (called "bracket")

$$(f,g) \mapsto \{f,g\} \in C$$

such that

$$\{f \cdot g, h\} = f \cdot \{g,h\} + g \cdot \{f,h\}.$$

DEFINITION 2.4. We call *integral* of a Hamiltonian system a function f such that $\dot{f} = 0$.

LEMMA 2.5. *The centralizer $Z(Q)$ of any set Q of elements of a Poisson algebra C is a Poisson algebra. In particular, for $C = C^\infty(M)$ the centralizer of H is exactly the collection of all integrals of the Hamiltonian system generated by H, and therefore this collection is a Poisson algebra.*

We describe now some examples.

EXAMPLE 2.6. For any non-degenerate Poisson structure we may choose local coordinates $(y^1, \dots, y^m)$ such that $m = 2k$ and

$$\omega^{ij} = \begin{pmatrix} 0 & 1_k \\ -1_k & 0 \end{pmatrix}.$$

EXAMPLE 2.7. For any degenerate Poisson structure with constant rank we may choose local coordinates $(y^1, \dots, y^m)$ such that $m = 2k + s$ and

$$\omega^{ij} = \begin{pmatrix} 0 & 1_k & 0 \\ -1_k & 0 & 0 \\ 0 & 0 & 0_s \end{pmatrix}.$$

EXAMPLE 2.8. Let us consider a Poisson structure ω^{ij} whose coefficients are linear functions of some coordinates $(y^1, \dots, y^m)$:

$$\omega^{ij} = C^{ij}_k y^k, \qquad C^{ij}_k = \text{const}.$$

Remark that

$$\{y^i, y^j\} = C^{ij}_k y^k;$$

therefore the collection of all linear functions is a Lie algebra which is finite-dimensional for the finite-dimensional manifold M; it is much smaller than the whole algebra $C^\infty(M)$ in any case ("Lie-Poisson bracket").

The annihilator of this bracket is exactly the collection of "Casimirs": it is generated by polynomials in the variables (y^i) which generate also the centre of the enveloping associative algebra $U(L)$ of L after a proper ordering of the variables (this is a non-obvious theorem). This bracket has been invented by

Sophus Lie about 100 years ago and later rediscovered by F. Berezin in 1960; it has been seriously used by Kirillov and Kostant in representation theory —see [7] and [12].

EXAMPLE 2.9. (For this and the next example see the survey [21]: they are special cases of Example 2.8.) Let L be a semisimple Lie algebra with a euclidean Killing form, $b_{ij} = \delta_{ij}$ and $M = L^* = L$. We consider the Poisson structure given as in Example 2.8 by $\omega^{ij} = c_k^{ij} y^k$, where the c_k^{ij}'s are the structure constants of L. For any diagonal quadratic Hamiltonian function

$$2H(y) = \sum_i q_i (y^i)^2$$

the corresponding Hamiltonian system admits the following "Euler form": for any element Y in the Lie algebra L, $Y = \sum_i a_i y^i$, we define the vector $\Omega = \sum_i q_i a_i y^i$ so that our Hamiltonian system has the form:

$$\dot{Y} = [Y, \Omega],$$
$$Y \in L, \quad \Omega \in L^* = L.$$

(This can be checked by elementary calculation.)

Suppose $L = so_n$ is the Lie algebra of the orthogonal group (i.e. the algebra of all skew-symmetric n-matrices). In this case the index (i) above is exactly the pair

$$i = (\alpha, \beta), \quad \alpha < \beta, \quad \alpha, \beta = 1, \dots, n, \quad m = n(n-1)/2.$$

By [1] the generalized "rigid body" system corresponds to the case:

$$q_i = q_{(\alpha,\beta)} = q_\alpha + q_\beta, \quad q_\alpha \geq 0.$$

More generally, let two collections of numbers

$$a_1, \dots, a_n, \quad b_1, \dots, b_n$$

be given in such a way that

$$q_i = q_{(\alpha,\beta)} = \frac{a_\alpha - a_\beta}{b_\alpha - b_\beta}, \quad \alpha < \beta.$$

The Euler system in this case admits the following so-called "λ-representation" (analogous to the one constructed by the author in 1974 —see [20]— for the finite-gap solutions of KdV and the finite-gap potentials of the Schrödinger operator):

$$\partial_t (Y - \lambda U) = [Y - \lambda U, \Omega - \lambda V].$$

Here Y and Ω are skew-symmetric matrices and

$$U = \operatorname{diag}(a_1, \dots, a_n), \qquad V = \operatorname{diag}(b_1, \dots, b_n)$$

(see Manakov's result of 1976, in [7]).

The collection of conservation laws (integrals) might be obtained from the coefficients of the algebraic curve Γ:

$$\Gamma : \det(Y - \lambda U - \mu 1) = P(\lambda, \mu) = 0.$$

We shall return to this type of examples later. Remark that a Riemann surface already appeared here.

EXAMPLE 2.10. Consider now the Lie algebra L of the group E_3 (the isometry group of euclidean 3-space $\mathbb{R}^3$). The Lie algebra L is 6-dimensional: it has a set of generators $\{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ satisfying the relations:

$$[\tilde{M}_i, \tilde{M}_j] = \varepsilon_{ijk}\tilde{M}_k, \;\; \varepsilon_{ijk} = \pm 1,$$
$$[\tilde{M}_i, \tilde{p}_j] = \varepsilon_{ijk}\tilde{p}_k,$$
$$[\tilde{p}_i, \tilde{p}_j] = 0.$$

We set $M = L^*$ and we denote by M_i and p_i the coordinates along $\tilde{M}_i$ and $\tilde{p}_i$ respectively. The elements of L determine obviously linear functions on L^*. There are exactly two independent quadratic functions

$$f_1 = p^2 = \sum_i p_i^2$$
$$f_2 = ps = \sum_i M_i p_i$$

such that

$$\{f_\alpha, C^\infty(M)\} = 0, \qquad \alpha = 1, 2$$

with respect to the Lie-Poisson bracket defined as above (Lie-Poisson-Berezin-Kirillov-Kostant bracket). We shall consider later the Hamiltonians

(a) $$2H = \sum a_i M_i^2 + \sum b_{ij}(p_i M_j + M_i p_j) + \sum c_{ij} p_i p_j$$

(b) $$2H = \sum a_i M_i^2 + 2W(l^i p_i)$$

(in case b we have $p^2 = 1$).

DEFINITION 2.11. A Hamiltonian system is called *completely integrable in the sense of Liouville* if it admits a "large enough" family of independent integrals which are in involution (i.e. have pairwise Poisson brackets equal to zero), where "large enough" means exactly $(\dim M)/2$ for non-degenerate Poisson structures (symplectic manifolds) or $k+s$ if $\dim M = 2k+s$ and the rank of the Poisson tensor (ω^{ij}) is equal to $2k$. Let us fix for the sequel a completely integrable Hamiltonian system and a family $\{f_1, \ldots, f_{k+s}\}$ of integrals as in the definition. The gradients of these integrals are linearly independent at the generic point. In any case the rank of the gradients' matrix (i.e. the number of linearly independent gradients of the integrals) depends locally on a collection of constants c_j only, because it is an orbit of the local commutative group generated by the Hamiltonian vector fields with Hamiltonian functions f_j.

Therefore the generic level surface:

$$\begin{cases} f_1 = c_1 \\ \cdots \\ f_{k+s} = c_{k+s} \end{cases}$$

(where the c_i's are constants) is a k-dimensional non-singular manifold N^k in M.

THEOREM 2.12. *If the Hamiltonian flows above are globally well-defined, the manifold N^k is the factor of $\mathbb{R}^k$ by a lattice:*

$$N^k = \mathbb{R}^k/\Gamma.$$

On N^k there are natural linear coordinates $(\varphi_1, \ldots, \varphi_k)$ generated by the commuting Hamiltonian vector fields above such that $\dot{\varphi}_i = \text{const}$.

COROLLARY 2.13. *If the level surface N^k is compact then it is diffeomorphic to the torus $T^k = (S^1)^k$.*

Let us assume now that $s = 0$, i.e. that the tensor (ω^{ij}) is invertible. As usual we denote by Ω the (closed) 2-form with local expression

$$\sum_{i<j} \omega_{ij} dy^i \wedge dy^j$$

where by definition $\omega^{il}\omega_{lj} = \delta^i_j$. Let us consider a compact level surface N^k. In a small neighborhood $U(N^k)$ in M we may introduce an important 1-form ω such that

$$d\omega = \Omega$$

(this is because we have that $\Omega = 0$ on N^k that it is homologous to zero in a small neighborhood).

An important theorem of Liouville-Arnold-Iost states that in the domain $U(N^k)$ there exist "action-angle" coordinates

$$(\varphi_1, \ldots, \varphi_k, J_1, \ldots, J_k)$$