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Cambridge University Press
978-0-521-09590-7 - Calculus: Basic Concepts and Applications
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Prerequisites

0

This chapter begins with a rapid review of elementary arithmetic and algebra, emphasizing only those techniques essential to an understanding of the calculus. No attempt is made to provide a complete logical development of the subject.

0.1 Fundamental operations; parentheses

We begin with a brief statement of familiar properties of the numbers of arithmetic.

It makes no difference in what order we add numbers: $3+4=7$ and $4+3=7$, and, in general, for a and b any numbers,

$$a+b=b+a. \tag{1}$$

Likewise, the way in which numbers are grouped for addition does not affect the result: $3+(4+5)=3+9=12$, and $(3+4)+5=7+5=12$. In general,

$$a+(b+c)=(a+b)+c. \tag{2}$$

Multiplication of natural (i.e., counting) numbers may be thought of as repeated addition. Instead of $4+4+4$, we write $3 \cdot 4$, and for $3+3+3+3$, we write $4 \cdot 3$. But both are equal to 12, and, in general,

$$a \cdot b=b \cdot a. \tag{3}$$

As with addition, the way in which numbers are grouped for multiplication does not matter: $3 \cdot (4 \cdot 5)=3 \cdot 20=60$, and $(3 \cdot 4) \cdot 5=12 \cdot 5=60$. In general,

$$a \cdot (b \cdot c)=(a \cdot b) \cdot c. \tag{4}$$

Adding two numbers and multiplying the result by a third number gives the same result as multiplying each of the first two by the third and then adding. For example, $3 \cdot (4+5)=3 \cdot 9=27$, and $3 \cdot 4+3 \cdot 5=12+15=27$. In general,

$$a \cdot (b+c)=(a \cdot b)+(a \cdot c). \tag{5}$$

Experience with sets of objects makes properties (1) through (5) intuitively clear for the natural numbers. As the number system is extended to include fractions, negative numbers, and so forth, definitions are so chosen that these properties hold for them also. Remember that the letters in algebra stand for numbers; hence, these properties are basic to all the manipulations of algebra.

Parentheses, and other grouping symbols such as brackets, [ ], and braces, { }, are essentially punctuation marks. Indicated operations inside parentheses are to be thought of as performed first. In equations (2) and (4) they are used to indicate special ways of looking at the expression. If we care only about the result, not how it is obtained, equation (2) says that we can omit the parentheses and write $a + b + c$ without ambiguity. Similarly, in (4), $a \cdot b \cdot c$ represents the same number whichever way of associating factors is chosen. In (5), the situation is different. Writing $a \cdot b + c$ gives no indication, without some further agreement, whether this means $a \cdot (b + c)$ or $(a \cdot b) + c$. But these are different: $3 \cdot (4+5) = 3 \cdot 9 = 27$, and $(3 \cdot 4)+5 = 12+5 = 17$. The universal convention is to choose the second. That is:

Unless there is notation to the contrary, multiplications (and divisions) are performed before additions (and subtractions).

The convention permits removal of parentheses on the right side of (5): $a \cdot (b + c) = a \cdot b + a \cdot c$.

So far we have used the dot to indicate multiplication. When there is no ambiguity, we can omit the dot. Obviously, 24 and $2 \cdot 4$ have different meanings, but $2 \cdot x$ can be written as $2x$ and $a \cdot b$ as ab. Equation (4) can be written

$$(ab)c = a(bc), \tag{4'}$$

and (5) as

$$a(b + c) = ab + ac. \tag{5'}$$

Property (5) can be used to "expand" $(a + b)(p + q)$ as follows:

$$(a + b)(p + q) = a(p + q) + b(p + q) = ap + aq + bp + bq. \tag{6}$$

If the two factors are alike, we use the shorter notation $(a + b)^2$ for $(a + b)(a + b)$, and likewise $(a + b)^3$ for $(a + b)(a + b)(a + b)$, and so forth. Then, as a special case of (6), we have

$$(a + b)^2 = a^2 + 2ab + b^2.$$

The difference $a - b$ is defined as the number d such that $a = b + d$, and the quotient $a \div b$, also written a/b, as the number q such that $a = bq$. Properties such as the following can be understood intuitively for the natural numbers by dealing with sets of objects and can be proved formally on the basis of these definitions and properties (1)–(5):

$$a-(b+c)=a-b-c,$$
$$a-(b-c)=a-b+c,$$
$$a(b-c)=ab-ac,$$
$$a\div(bc)=(a\div b)\div c.$$

Calculators are designed to make it easy and natural to follow conventional arithmetic and algebraic usage. However, there are variations in the way different calculators work, and the beginner on any calculator must study its characteristics carefully. With practice, one soon learns to observe the standard conventions just as instinctively as in hand calculation.

PROBLEMS

1. Evaluate each of the following expressions:
   (a) $5+3\cdot 5$ (b) $2+\dfrac{12}{3}$ (c) $\dfrac{3+7}{1+4}$ (d) $3\cdot 96+3\cdot 4$ (e) $10-2\cdot 5$
   (f) $(10-2)\cdot 5$ (g) $(6\div 2)\div 3$ (h) $6\div(2\div 3)$ (i) $12-(4+2)$
   (j) $(9+4)-(7-3)$ (k) $2\cdot[7+3-(2\cdot 4)]$ (l) $(4+6\cdot 4)\div 7$
2. Simplify each of the following expressions:
   (a) $(a+p)(p+q)-p^2-pq$ (b) $(c+2d)^2-4cd$
   (c) $(a+b)^2-2(a^2+ab)$ (d) $x^2-xy+y^2-x(x-y)$
3. ○ Expand each of the following expressions, and keep your results for future reference:
   (a) $(x+h)^3$ (b) $(x+h)^4$ (c) $(x+h)^5$
4. Evaluate the expressions in each of the following pairs.
   (a) $5\cdot 10-8,\quad 5(10-8)$
   (b) $17-6+5,\quad 17-(6+5)$
   (c) $8+4\div 2,\quad (8+4)\div 2$
   (d) $2+3^2,\quad (2+3)^2$
   (e) $x\cdot x+y,\quad x(x+y)$
   (f) $sr-rr+s,\quad s[r-r(r+s)]$

0.2 Zero and negatives

The number 0 is defined by the property $a+0=a$ for all numbers a. We have, from this definition and (5),

$$a\cdot 0=0 \quad \text{for all } a. \tag{7}$$

An important consequence follows:

$$\text{If } ab=0, \quad \text{then } a=0 \quad \text{or} \quad b=0. \tag{8}$$

Division by 0 is impossible. For suppose that $2\div 0=q$. Then, by the definition of division, $q\cdot 0=2$. But, by (7), there is no such number q. The same argument applies for $a \div 0$, a being any number other than 0. Now suppose $0 \div 0 = q$. Again, by the definition of division, $0 \cdot q = 0$. Here q could be anything: $\frac{1}{2}$, 0, 1, 100, and 9999 all satisfy this condition; there is no way of picking out one number as the "answer."

Dividing 0 by any number different from 0 gives no trouble: Suppose that $0 \div a = q$, with $a \neq 0$. Then $aq = 0$ and $q = 0$, by (8).

The negative of a number a is the number x such that $a + x = 0$, and it is denoted by $-a$. The negatives of the positive numbers are the negative numbers, and the negatives of the negative numbers are the positive numbers. Positive numbers are greater than 0, negative numbers less than 0. Note that 0 is neither positive nor negative and that the negative of 0 is 0. The natural numbers, their negatives, and 0 compose the set of integers. The following familiar results follow from this definition and the properties (1)–(5) in Section 0.1.

For all p, q, $p + (-q) = p - q$. (Note that the minus sign plays two roles – as a label for the negative of a number and as the symbol for subtraction. This equation means that $p - q$ can be thought of either as indicating the subtraction of q from p or as indicating the addition of p and $-q$.) Similarly, for all p, q, r,

$$p - (-q) = p + q,$$

$$-(-p) = p,$$

$$-p = (-1)p,$$

$$(-p) \cdot q = -(pq),$$

$$(-p)(-q) = pq,$$

$$\frac{p}{-q} = \frac{-p}{q} = -\left(\frac{p}{q}\right),$$

$$\frac{-p}{-q} = \frac{p}{q},$$

$$r(p - q) = rp - rq.$$

PROBLEMS

1. Evaluate, if possible, for $x = 0$, $x = 1$, and $x = 3$:
   (a) $\dfrac{x^2 - 9}{x + 3}$ (b) $\dfrac{x^2 + 9}{x - 3}$ (c) $\dfrac{x^2 - 9}{x - 3}$ (d) $\dfrac{x}{x^2 - 4x + 3}$
2. Evaluate each of the following expressions:
   (a) $5 + (-8)$ (b) $5 - (-8)$ (c) $4 \cdot 106 - 4 \cdot 6$
   (d) $12 - \dfrac{10}{-2}$ (e) $12 + \dfrac{-10}{2}$ (f) $12 - \dfrac{-10}{-2}$
   (g) $\dfrac{12 - 10}{-2}$ (h) $\dfrac{12 + (-10)}{2}$ (i) $\dfrac{12 - (-10)}{-2}$

(j) $\dfrac{16-10}{4-1}$ (k) $\dfrac{-16-10}{1-14}$ (l) $\dfrac{1-3\cdot 5}{5-(-2)}$

(m) $\dfrac{56-8(7+5)}{2(1-6)}$

3. Simplify each of the following expressions:
(a) $(a+b)(p+q)-(a+b)p$ (b) $x(u+v)-2xu$
(c) $(x-y)^2+2xy$ (d) $(c-d)-(a-d)-(c-a)$
(e) $ab-[cd-(ef-ab)]$ (f) $\dfrac{ac-bc}{a-(a+c)}$

○ 4. Simplify each of the following expressions:
(a) $(x-y)(x+y)$ (b) $(x-y)(x^2+xy+y^2)$
(c) $(x-y)(x^3+x^2y+xy^2+y^3)$

5. Expand each of the following expressions:
(a) $(x-h)^2$ (b) $(x-h)^3$ (c) $(x-h)^4$ (d) $(x-h)^5$

0.3 Fractions and rational numbers

A rational number is one that can be expressed as the quotient of two integers; that is, as a fraction with numerator and denominator integers. Every integer satisfies this definition because it can be expressed (in many ways) as such a quotient; for example, $3 = 3 \div 1 = 6 \div 2 = 15 \div 5$, and so on. The way we read a common fraction like two-thirds indicates that we are thinking of it as $2\cdot(\frac{1}{3})$. But $3(2)(\frac{1}{3}) = 2(3)(\frac{1}{3}) = 2\cdot 1 = 2$; hence, it satisfies the definition of the quotient $2 \div 3$. We can choose whichever interpretation of $\frac{2}{3}$ suits us.

Common sense assures us that $2\cdot(\frac{1}{3}) = 4\cdot(\frac{1}{6})$ – twice as many parts, each half as big. That is, $\frac{2}{3} = \frac{2\cdot 2}{3\cdot 2} = \frac{4}{6}$. In general, for $k \neq 0$,

$$\frac{a}{b} = \frac{ka}{kb}. \tag{9}$$

(In this and the formulas that follow, assume that the denominators of the given fractions are not 0.) We can use this property to reduce a fraction to "lower terms" – $\frac{10}{15} = \frac{2\cdot 5}{3\cdot 5} = \frac{2}{3}$ – or to change to "higher terms" – $\frac{3}{25} = \frac{3\cdot 4}{25\cdot 4} = \frac{12}{100} = 0.12$.

If a decimal terminates, it can be written as a fraction whose denominator is a power of 10 (e.g., $0.12 = \frac{12}{100}$), and hence it is rational. The converse is *not* true; for example, $\frac{1}{3} = 0.333\ldots$, continued indefinitely.

Multiplying simple fractions like $\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$, and $\frac{5}{2}\cdot\frac{7}{3} = (5\cdot\frac{1}{2})(7\cdot\frac{1}{3}) = (5\cdot 7)(\frac{1}{2}\cdot\frac{1}{3}) = \frac{35}{6}$, leads to the general rule

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}. \tag{10}$$

The rule for dividing fractions can be obtained by applying (9) to

$$\frac{a/b}{c/d} = \frac{(a/b)\cdot(d/c)}{(c/d)\cdot(d/c)} = \frac{ad/bc}{1} = \frac{ad}{bc}.$$

In other words:

To divide by a fraction, invert the divisor and multiply.

To see how to add fractions, we look again at cases in which it is easy to see what the answer must be: $\frac{1}{3}+\frac{2}{3}=\frac{3}{3}=1$; $\frac{4}{5}+\frac{3}{5}=(4+3)\cdot\frac{1}{5}=\frac{7}{5}$. Clearly, if the fractions have the same denominator, the numerator of the sum is the sum of the numerators, and the denominator is that common denominator. But by the use of (9) we can always express each of the fractions so that they do have a common denominator: For example, $\frac{1}{2}+\frac{4}{3}=\frac{1\cdot 3}{2\cdot 3}+\frac{4\cdot 2}{3\cdot 2}=\frac{3}{6}+\frac{8}{6}=\frac{11}{6}$. In general,

$$\frac{a}{b}+\frac{c}{d}=\frac{a\cdot d}{b\cdot d}+\frac{b\cdot c}{b\cdot d}=\frac{ad+bc}{bd}. \tag{11}$$

Whereas (11) always gives the correct result, that result can sometimes be obtained more easily: For example, $\frac{5}{6}+\frac{3}{4}=\frac{10}{12}+\frac{9}{12}=\frac{19}{12}$, whereas (11) would have us say $\frac{20}{24}+\frac{18}{24}=\frac{38}{24}$, which reduces to $\frac{19}{12}$, the same result, of course. The work will be simplest if we use as the denominator the least number that contains both denominators as factors; to see what this is in a less obvious case, we write the denominators in factored form.

Example 1

$$\frac{7}{60}+\frac{5}{72}=\frac{7}{2^2\cdot 3\cdot 5}+\frac{5}{2^3\cdot 3^2}.$$

The least common denominator is $2^3\cdot 3^2\cdot 5$. We have then

$$\frac{7\cdot 2\cdot 3}{2^2\cdot 3\cdot 5\cdot 2\cdot 3}+\frac{5\cdot 5}{2^3\cdot 3^2\cdot 5}=\frac{42}{360}+\frac{25}{360}=\frac{67}{360},$$

which is in lowest terms. Rule (11) would give $\frac{7}{60}+\frac{5}{72}=\frac{804}{4320}$, which reduces to $\frac{67}{360}$.

Obviously the arithmetic needed involves larger numbers than the first method. We could replace (11) by the following rule, but it is awkward to write it as a formula:

> To add fractions (with minimum labor), change each fraction to one whose denominator is the least common denominator for all the fractions; then add the numerators and set that result over the common denominator. (11′)

Example 2

$$\frac{1}{3+h}-\frac{1}{3}=\frac{1\cdot 3}{(3+h)\cdot 3}-\frac{1\cdot(3+h)}{3(3+h)}=\frac{3-(3+h)}{3(3+h)}$$
$$=\frac{3-3-h}{3(3+h)}=\frac{-h}{3(3+h)}.$$

Cambridge University Press
978-0-521-09590-7 - Calculus: Basic Concepts and Applications
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Example 3

$$\frac{1}{2x(x+h)} - \frac{1}{2x^2}.$$

The least common denominator is $2x^2(x+h)$.

$$\frac{1}{2x(x+h)} - \frac{1}{2x^2} = \frac{1\cdot x}{2x(x+h)\cdot x} - \frac{1\cdot(x+h)}{2x^2(x+h)} = \frac{x-x-h}{2x^2(x+h)}$$

$$= \frac{-h}{2x^2(x+h)}.$$

In the preceding examples there is no point in multiplying out in the denominators (except possibly in the last step for some purposes, especially in numerical examples); on the other hand, it *is* necessary to multiply out in the numerators in order to combine like terms.

In the rational numbers we have a set closed under the operations of addition, subtraction, multiplication, and division (i.e., combining any two numbers of the set by any of these operations gives again a member of the set), with the single exception of division by 0. Although we have by no means done so, it can be shown that properties (1) through (5) hold for the rational numbers.

PROBLEMS

1. Reduce each of the following fractions to a simpler form, if possible:
   (a) $\frac{6}{15}$ (b) $\frac{100}{24}$ (c) $\frac{504}{108} = \frac{2^3\cdot 3^2\cdot 7}{2^2\cdot 3^3}$ (d) $\frac{90}{675}$ (e) $\frac{12}{3/4}$
   (f) $\frac{12/3}{4}$ (g) $\frac{36h}{6(6+h)}$ (h) $\frac{2x^2}{10x(x+2)}$ (i) $\frac{3y}{(y+3)/y^2}$
   (j) $\frac{3y/(y+3)}{y^2}$
2. Perform the indicated additions and subtractions:
   (a) $\frac{5}{6}+\frac{2}{7}$ (b) $\frac{11}{30}-\frac{2}{21}$ (c) $\frac{5b}{a^2+ab}+\frac{5}{a+b}$
   (d) $\frac{y}{x+y}-\frac{z}{x+z}$ (e) $\frac{3x}{5}+\frac{7}{2x}$ (f) $a+\frac{b}{c}$
3. ○ Simplify each of the following expressions:
   (a) $\left(\frac{1}{4+h}-\frac{1}{4}\right)\Big/h$ (b) $\left(\frac{3}{(x+h)^2}-\frac{3}{x^2}\right)\Big/h$
   (c) $\left(\frac{1}{(x+h)^3}-\frac{1}{x^3}\right)\Big/h$

0.4 Integral exponents

We have assumed familiarity with the definition

$$x^m = x \cdot x \cdot \ldots \cdot x \quad (m \text{ factors}), \quad m \text{ a positive integer.}$$

This definition leads immediately to the following results, where m and n are positive integers:

$$(xy)^m = x^m y^m \tag{12}$$

$$x^m x^n = x^{m+n} \tag{13}$$

$$(x^m)^n = x^{mn} \tag{14}$$

$$\frac{x^m}{x^n} = x^{m-n}, \quad \text{if } m > n \quad (\text{read } m \text{ greater than } n), \quad \text{and } x \neq 0. \tag{15}$$

We define x^m for zero and negative m in such a way that (12)–(15) hold. If we ignore the restriction $m > n$ in (15), we have, for example, $x^4/x^7 = x^{4-7} = x^{-3}$, $x \neq 0$. On the other hand, $x^4/x^7 = 1/x^{7-4} = 1/x^3$, $x \neq 0$. In general, we define

$$x^{-m} = \frac{1}{x^m}, \quad x \neq 0, \quad m \text{ a positive integer.} \tag{16}$$

Similarly, if $m = n$, $x^m/x^m = x^{m-m} = x^0$, $x \neq 0$. But $x^m/x^m = 1$, $x \neq 0$, and $0^m/0^m = 0/0$ is a meaningless symbol. Hence, we define

$$x^0 = 1, \quad x \neq 0. \tag{17}$$

Note that 0^0 is undefined.

PROBLEMS

Simplify the following expressions, writing each of them without negative exponents.

1. $x + x^{-1}$
2. $(x + x^{-1})^2$
3. $\dfrac{x^2 - 1}{2x^{-1}}$
4. $(7x)^0 + \dfrac{7}{x^0}$
5. $\dfrac{3x}{(2x)^2 - 1} - \dfrac{1}{2x - 1}$
6. $\dfrac{x^{-2}}{2^{-1}x - 1}$
7. $\dfrac{x + y}{x^{-1} + y^{-1}}$
8. $\dfrac{a^{-2} - b^{-2}}{a - b}$
9. $(x^{-1} + y^{-1})^{-2}$
10. $\left(1 + \dfrac{1}{x}\right)^0$
11. $\dfrac{a^{-2}}{2b^{-2}}$
12. $k(r + k)^{-2} - kr^{-2}$
13. $\dfrac{(a^3 \cdot a^2)^4}{(ab)^7}$
14. $(rs)^{2x} r^{-x} s^{-3x}$
15. $(w^{-2} + w^{-3})^2$
16. $(w^{-2} + w^{-3})^{-2}$

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0.5 Radicals, fractional exponents, and real numbers

Recall now the familiar notation using radicals to denote roots of numbers. By definition,

$$(\sqrt[q]{x})^q = x, \quad \text{where } q \text{ is a positive integer.}$$

For example, $(\sqrt{5})^2 = 5$, $(\sqrt{9})^2 = 9$. Now, both $3^2 = 9$ and $(-3)^2 = 9$. It is agreed that $\sqrt{9} = 3$, the positive root only; then $-3 = -\sqrt{9}$. (There is a discrepancy here, which may be confusing, between the way we read $\sqrt{x}$ and the precise definition of the symbol; strictly speaking, we should say "the positive square root of x" instead of simply "the square root of x," as we usually do.) Likewise, $\sqrt{5}$ stands for the positive root only. For q any even number and $x > 0$, we have a similar situation; for example, $\sqrt[4]{16} = 2$, not -2. For q even and $x < 0$, there is no real root; nevertheless, we shall later find meanings for such expressions. For q odd, there is exactly one real root, and so we need no such convention; for example, $\sqrt[3]{8} = 2$, $\sqrt[3]{-8} = -2$.

It is easily shown that

$$\sqrt[q]{ab} = \sqrt[q]{a} \cdot \sqrt[q]{b} \quad \text{and} \quad \sqrt[q]{\frac{a}{b}} = \frac{\sqrt[q]{a}}{\sqrt[q]{b}}. \tag{18}$$

This gives a means of simplifying radical expressions, or changing them to more convenient forms (e.g., without radicals in denominators). Examples:

$$\sqrt[3]{54} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}.$$

$$\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{\sqrt{4}} = \frac{\sqrt{6}}{2}.$$

$$\sqrt{5a^2 + 10ab + 5b^2} = \sqrt{5(a^2 + 2ab + b^2)}$$

$$= \sqrt{5} \cdot \sqrt{(a+b)^2} = \sqrt{5}\,(a+b), \quad \text{if } a + b \geq 0.$$

$$\sqrt[4]{16a^4 + 16b^4} = \sqrt[4]{16} \cdot \sqrt[4]{a^4 + b^4} = 2\sqrt[4]{a^4 + b^4}.$$

$$\frac{\sqrt{x+y}}{\sqrt{x-y}} = \frac{\sqrt{x+y}}{\sqrt{x-y}} \cdot \frac{\sqrt{x-y}}{\sqrt{x-y}} = \frac{\sqrt{x^2-y^2}}{x-y}, \quad \text{if } x - y > 0.$$

We return now to exponents. If we apply (14), disregarding the restriction that m be an integer, we have $(x^{1/q})^q = x^{q/q} = x$, and so we define

$$x^{1/q} = \sqrt[q]{x}, \quad q \text{ a positive integer.} \tag{19}$$

This means that everything we have said about radical expressions can be stated in terms of fractional exponents. For example, (18) becomes

$$(ab)^{1/q} = a^{1/q}b^{1/q} \quad \text{and} \quad \left(\frac{a}{b}\right)^{1/q} = \frac{a^{1/q}}{b^{1/q}}.$$

Now, using (14) and neglecting the restriction that m be an integer, we have $x^{p/q}=(x^{1/q})^p$, p and q integers, $q>0$. For example, $8^{2/3}=(8^{1/3})^2=2^2=4$. If $x>0$, then also $x^{p/q}=(x^p)^{1/q}$; if $x<0$ and if p/q is not in lowest terms, this may lead to error. For example, $(-8)^{2/6}=(-8)^{1/3}=-2$, but $[(-8)^2]^{1/6}=(64)^{1/6}=2$, not -2. We therefore take as the definition of $x^{p/q}$,

$$x^{p/q}=(x^{1/q})^p, \quad p \text{ and } q \text{ integers without a common factor, } q>0. \quad (20)$$

With definitions (16), (17), (19), and (20) it can be shown that properties (12) through (15) hold for m and n any rational numbers. The only restrictions that are retained are that $x \neq 0$ in (15) and that 0^0 remains undefined.

In the preceding discussion we mentioned some symbols (e.g., $\sqrt{3}, \sqrt{5}, \sqrt[3]{2}$) that have no meaning in the rational number system. That is, it can be proved that there is no rational number whose square is 3, and so forth. The set of real numbers can be defined as the set of all decimal representations, terminating and nonterminating. The rationals compose the subset with decimal representation, either terminating or periodic from some point on. (This is not hard to show.) All other real numbers are irrational. For example, the following numbers are all rational:

$\frac{51}{4}=12.75$,

$\frac{5}{3}=1.66\ldots$ (the dots mean "continued indefinitely"),

$\frac{1}{22}=0.04\dot{5}\dot{4}\ldots$ (the dots above 45 indicate the period),

$\frac{22}{7}$,

3.1416.

It can be shown that $\sqrt{2}$, $\sqrt[3]{9}$, and the number π are irrational.

The real numbers can be put into one-to-one correspondence with the points of a line, once a zero point, a unit point, and a positive direction have been chosen, so that numbers on the number line increase in the positive direction and decrease in the opposite (negative) direction.

Even in the set of real numbers, we have no number whose square is a negative number. Later we shall make one more extension of our number system which will remedy that lack.

PROBLEMS

1. Simplify:

(a) $\sqrt[4]{48}$ (b) $\sqrt[3]{\dfrac{4}{9}}$ (c) $\sqrt{a^3-2a^2b+ab^2}$

(d) $\sqrt[3]{8a^3-8b^3}$ (e) $\dfrac{5}{\sqrt{5(a+b)^3}}$