CHAPTER 1

PRELIMINARIES

1. Notations

If A is a set then $x \in A$ means x is an element of A. If A and B are sets then $A \subset B$ or $B \supset A$ means every element of A is an element of B. By $A \cap B$ we mean the set of all elements belonging to both A and B. By $A \cup B$ we mean the set of elements in either A or B (or both). The empty set (the set with no elements) will be denoted by $\varnothing$.

The open interval of real numbers x for which $a < x < b$ will often be denoted by (a, b). Similarly, $[a, b]$ will denote the closed interval $a \leqslant x \leqslant b$.

By a *neighborhood* of the real number x we mean any set which contains an interval $(x-\delta, x+\delta)$ for some $\delta > 0$. A neighborhood of x will often be denoted by N_x.

Some other definitions concerning point sets will be introduced (when needed) in § 14.

If θ is a complex number then $\bar{\theta}$ will denote the complex conjugate of θ.

We shall often abbreviate 'almost everywhere' as a.e.

It is convenient to denote in parentheses, to the right of a displayed statement, the set of values of the variable or variables for which the statement is true. For example,

$$\hat{f}(x) = 0 \quad (a \leqslant x \leqslant b)$$

means that $\hat{f}(x)$ is equal to zero for every x in the interval $[a, b]$.

2. Some integral theorems

We shall list here many important theorems on the Lebesgue and Riemann–Stieltjes integrals which are used in the text. This will permit the reader to review the theorems and, in addition, will permit us to refer to the theorems by number later on. For proofs of these results see [4] or [19] (Lebesgue integral) and [20] or [4] (Stieltjes integral).

Unless it is specifically stated to the contrary, we shall assume that all functions used are complex-valued and measurable.

2A. THEOREM (FATOU'S LEMMA). Let $f_1, f_2, \ldots$ be non-negative functions on $(-\infty, \infty)$. If

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

then
$$\int_{-\infty}^{\infty} f(x)\,dx \leqslant \varliminf_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

2B. THEOREM (LEBESGUE CONVERGENCE THEOREM). Let $f_1, f_2, \ldots$ be integrable on $(-\infty, \infty)$. If

$$|f_n(x)| \leqslant F(x) \text{ a.e.} \quad (-\infty < x < \infty;\ n = 1, 2, \ldots)$$

for some integrable F, and if

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

then f is integrable and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

In 2B it is not necessary that the family $\{f_n\}$ be denumerable. For example, the same theorem holds for a family $\{f_n\}$ where n runs through, say, all positive real numbers. See [12; 169].

2C. THEOREM (FUBINI). If the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy$$

is absolutely convergent, then

$$\int_{-\infty}^{\infty} f(x, y)\,dy$$

exists for almost all x and is an integrable function of x. Moreover

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x, y)\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy.$$

Similarly,

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy.$$

2D. THEOREM (TONELLI–HOBSON). If either of the iterated integrals

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x,y)\,dy, \quad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)\,dx$$

is absolutely convergent, then so is the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

and hence (2C) all three integrals are equal.

2E. THEOREM. If f is integrable on $[-R, R]$ for every $R > 0$ then

(*) $$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x+t)-f(x)|\,dt = 0 \text{ a.e.} \quad (-\infty < x < \infty).$$

The set of points x for which (*) holds is called the Lebesgue set for f. The Lebesgue set clearly contains all points at which f is continuous.

2F. DEFINITION (THE CLASS L^p). Suppose $1 \leqslant p < \infty$. The function f on $(-\infty, \infty)$ is said to be of class L^p (written $f \in L^p$) if $\int_{-\infty}^{\infty} |f(x)|^p\,dx < \infty$. If $f \in L^p$ then $\|f\|_p$ is defined to be

$$\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}.$$

The symbol $\|f\|_p$ is read as the L^p norm of f.

2G. REMARK. If two functions in L^p coincide except on a set of measure zero, they are considered to represent the same element of L^p. Thus, the elements of L^p are really *classes* of functions, the functions in any one class differing from one another only on sets of measure zero. As is customary we shall abuse language and speak of individual functions as elements of L^p.

2H. THEOREM. Let $f, f_1, f_2, \ldots$ be in L^p. If

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

and
$$\lim_{n\to\infty} \|f_n\|_p = \|f\|_p,$$

then
$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

This theorem is perhaps not so well known as the others. For a proof see [9; 34].

2I. THEOREM (L^p IS COMPLETE). Let $f_1, f_2, \ldots$ be in L^p. If

$$\lim_{m,n\to\infty} \|f_m - f_n\|_p = 0,$$

then there exists $f \in L^p$ such that

$$\lim_{n\to\infty} \|f - f_n\|_p = 0.$$

2J. THEOREM (CONTINUITY IN THE MEAN). If $f \in L^p$ then

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)|^p \, dx = 0.$$

2K. THEOREM. Let $f, f_1, f_2, \ldots$ be in L^p. If

$$\lim_{n\to\infty} \|f_n - f\|_p = 0,$$

and if for some g

$$\lim_{n\to\infty} f_n(x) = g(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

then
$$f(x) = g(x) \text{ a.e.} \quad (-\infty < x < \infty).$$

(That is, if f is the limit in the mean of f_n, and g is the pointwise limit of f_n, then $f = g$.)

2L. THEOREM. If $f, g \in L^p$ then

$$\text{(i)} \quad \|f+g\|_p \leqslant \|f\|_p + \|g\|_p,$$

and
$$\text{(ii)} \quad \big| \|f\|_p - \|g\|_p \big| \leqslant \|f-g\|_p.$$

2M. THEOREM (SCHWARZ INEQUALITY). If $f, g \in L^2$ then $fg \in L^1$ and

$$\|fg\|_1 \leqslant \|f\|_2 \cdot \|g\|_2.$$

That is,

$$\left(\int_{-\infty}^{\infty} |f(x)\, g(x)| \, dx\right)^2 \leqslant \int_{-\infty}^{\infty} |f(x)|^2 \, dx \, . \int_{-\infty}^{\infty} |g(x)|^2 \, dx.$$

2N. THEOREM. Let $f, f_1, f_2, \ldots$ be in L^2. If

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0$$

then, for any $g \in L^2$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

The last three theorems are concerned with Riemann and Riemann–Stieltjes integrals.

2P. THEOREM (MEAN VALUE THEOREM). If f is continuous on $[a, b]$ and if α is non-decreasing and bounded on $[a, b]$, then, for some c in $[a, b]$,

$$\int_a^b f(x)\, \alpha(x)\, dx = \alpha(a^+) \int_a^c f(x)\, dx + \alpha(b^-) \int_c^b f(x)\, dx.$$

2Q. THEOREM. Let $\Phi_1, \Phi_2, \ldots$ be non-decreasing functions on $(-\infty, \infty)$ such that $\Phi_n(t) = \frac{1}{2}[\Phi_n(t^+) + \Phi_n(t^-)]$ for all n and t. If, for some $M > 0$,

$$|\Phi_n(t)| \leqslant M \quad (-\infty < t < \infty;\ n = 1, 2, \ldots),$$

then there exists a sequence $n_1, n_2, \ldots$ and a non-decreasing function Φ such that

$$\lim_{k\to\infty} \Phi_{n_k}(t) = \Phi(t) \quad (-\infty < t < \infty).$$

Moreover, if f is continuous on $(-\infty, \infty)$ and $\lim_{t\to\pm\infty} f(t) = 0$, then

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f(t)\, d\Phi_{n_k}(t) = \int_{-\infty}^{\infty} f(t)\, d\Phi(t).$$

2R. THEOREM. Let $f, f_1, f_2, \ldots$ be continuous on $(-\infty, \infty)$. If, for some $M > 0$,

$$|f_n(t)| \leqslant M \quad (-\infty < t < \infty;\ n = 1, 2, \ldots),$$

and if

$$\lim_{n\to\infty} f_n(t) = f(t) \quad (-\infty < t < \infty),$$

then for any α of bounded variation on $(-\infty, \infty)$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\, d\alpha(t) = \int_{-\infty}^{\infty} f(t)\, d\alpha(t).$$

CHAPTER 2

THE FOURIER TRANSFORM ON L^1

3. Definition of the Fourier transform

3 A. For each $f \in L^1$ the integral

$$\int_{-\infty}^{\infty} e^{ixt} f(t)\, dt$$

exists for all real x. We define the Fourier transform $\hat{f}$ of $f \in L^1$ by

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt \quad (-\infty < x < \infty).$$

Since $$|\hat{f}(x)| \leqslant \int_{-\infty}^{\infty} |e^{ixt} f(t)|\, dt = \int_{-\infty}^{\infty} |f(t)|\, dt = \|f\|_1,$$

we see that $\hat{f}$ is bounded on $(-\infty, \infty)$ and

$$\underset{-\infty < x < \infty}{\text{l.u.b.}} |\hat{f}(x)| \leqslant \|f\|_1. \tag{1}$$

Moreover, $\hat{f}$ is continuous on $(-\infty, \infty)$. To see this we have, for any real x and h,

$$\hat{f}(x+h) - \hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt}(e^{iht} - 1) f(t)\, dt,$$

so that $$|\hat{f}(x+h) - \hat{f}(x)| \leqslant \int_{-\infty}^{\infty} |e^{iht} - 1| \cdot |f(t)|\, dt. \tag{2}$$

The integrand on the right of (2) is not greater than $2|f(t)|$ and tends to zero as $h \to 0$. Hence, by the Lebesgue convergence theorem 2 B, the right side of (2) tends to zero as $h \to 0$. The left side of (2) must, therefore, also tend to zero as $h \to 0$, which shows that $\hat{f}$ is continuous at x.

3 B. THEOREM. If $f, f_1, f_2, \ldots$ are in L^1 and if $\|f_n - f\|_1 \to 0$ as $n \to \infty$, then

$$\lim_{n \to \infty} \hat{f}_n(x) = \hat{f}(x) \text{ uniformly} \quad (-\infty < x < \infty).$$

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PROOF. From (1) of 3A we have

$$\operatorname*{l.u.b.}_{-\infty<x<\infty} |\hat{f}_n(x)-\hat{f}(x)| \leqslant \|f_n-f\|_1$$

from which the theorem follows immediately.

In later sections we shall discuss translates $f(t+b)$ of a function $f(t)$. We record the next theorem for future reference.

3C. THEOREM. Fix the real numbers a and b. If $f \in L^1$ then the Fourier transform of $f(t+a)$ is $\hat{f}(x)\,e^{-iax}$. Also, $\hat{f}(x+b)$ is the Fourier transform of $e^{ibt}f(t)$. Thus, any translate of a Fourier transform of an L^1 function is again a Fourier transform.

PROOF. We have, by a change of variable,

$$\int_{-\infty}^{\infty} e^{ixt} f(t+a)\,dt = \int_{-\infty}^{\infty} e^{ix(t-a)} f(t)\,dt = e^{-iax}\hat{f}(x). \tag{1}$$

Also, $$\hat{f}(x+b) = \int_{-\infty}^{\infty} e^{i(x+b)t} f(t)\,dt = \int_{-\infty}^{\infty} e^{ixt}[e^{ibt}f(t)]\,dt. \tag{2}$$

The theorem follows from (1) and (2).

4. The Riemann–Lebesgue theorem

This famous theorem states that the Fourier transform of an L^1 function must vanish at $\pm\infty$. Specifically,

4A. THEOREM (RIEMANN–LEBESGUE). If $f \in L^1$ then

$$\lim_{x\to\pm\infty} \hat{f}(x) = \lim_{x\to\pm\infty} \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt = 0.$$

PROOF. Since $$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt, \tag{1}$$

then $$-\hat{f}(x) = \int_{-\infty}^{\infty} e^{ix[t+(\pi/x)]} f(t)\,dt = \int_{-\infty}^{\infty} e^{ixt} f\left(t-\frac{\pi}{x}\right) dt. \tag{2}$$

Subtracting (2) from (1) we obtain

$$2\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt}\left[f(t)-f\left(t-\frac{\pi}{x}\right)\right] dt.$$

Hence $$2|\hat{f}(x)| \leqslant \int_{-\infty}^{\infty} \left| f(t)-f\left(t-\frac{\pi}{x}\right)\right| dt. \tag{3}$$

But, since $f \in L^1$,

$$\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} \left| f(t) - f\left(t - \frac{\pi}{x}\right) \right| dt = 0 \tag{4}$$

by Theorem 2J. The theorem follows from (3) and (4).

4B. Corollary. If $f \in L^1$ then

$$\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \sin xt \, dt = \lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \cos xt \, dt = 0.$$

Proof. This follows from 4A and the identities

$$2i \sin \theta = e^{i\theta} - e^{-i\theta}, \quad 2 \cos \theta = e^{i\theta} + e^{-i\theta}.$$

4C. We have now shown that if $f \in L^1$ then $\hat{f}$ is continuous on $(-\infty, \infty)$ and $\lim_{x \to \pm\infty} \hat{f}(x) = 0$. It is natural to ask whether every function which is continuous on $(-\infty, \infty)$ and which vanishes at $\pm\infty$ is the Fourier transform of a function in L^1. The answer is 'no' as the following example attests.

Example. Let

$$g(x) = \frac{1}{\log x} \quad (x > e,)$$

$$= \frac{x}{e} \quad (0 \leqslant x \leqslant e),$$

$$= -g(-x) \quad (x < 0).$$

Then g is continuous on $(-\infty, \infty)$ and g vanishes at $\pm\infty$. We shall show, however, that g is *not* a Fourier transform. We first note that

$$\lim_{N \to \infty} \int_e^N \frac{g(x)}{x} dx = \lim_{N \to \infty} \int_e^N \frac{dx}{x \log x} = \lim_{N \to \infty} \log \log N = \infty. \tag{1}$$

Now suppose that there is an $f \in L^1$ such that $g = \hat{f}$. Then

$$g(x) = \int_{-\infty}^{\infty} e^{ixt} f(t) \, dt \quad (-\infty < x < \infty),$$

and, since $g(x) = -g(-x)$, we also have

$$g(x) = -\int_{-\infty}^{\infty} e^{-ixt} f(t) \, dt.$$

Adding the preceding two equations we obtain

$$2g(x) = 2i\int_{-\infty}^{\infty} f(t)\sin xt\,dt.$$

Hence $$g(x) = i\int_0^{\infty} f(t)\sin xt\,dt + i\int_{-\infty}^{0} f(t)\sin xt\,dt$$

$$= i\int_0^{\infty} f(t)\sin xt\,dt - i\int_0^{\infty} f(-t)\sin xt\,dt,$$

and finally, $$g(x) = \int_0^{\infty} F(t)\sin xt\,dt.$$

Here $F(t) = i[f(t) - f(-t)]$ so that $\int_0^{\infty} |F(t)|\,dt < \infty$. For any $N = 3, 4, 5, \ldots$ we have

$$\int_e^N \frac{g(x)}{x}\,dx = \int_e^N \frac{dx}{x}\int_0^{\infty} F(t)\sin xt\,dt.$$

Since $\int_0^{\infty} |F(t)|\,dt < \infty$ we may change the order of integration in the iterated integral on the right (by Theorem 2D) to obtain

$$\int_e^N \frac{g(x)}{x}\,dx = \int_0^{\infty} F(t)\,dt\int_e^N \frac{\sin xt}{x}\,dx,$$

$$\int_e^N \frac{g(x)}{x}\,dx = \int_0^{\infty} F(t)\,dt\int_{et}^{Nt} \frac{\sin x}{x}\,dx. \tag{2}$$

But $\left|\int_a^b \frac{\sin t}{t}\,dt\right|$ is bounded for all real a and b and, for each t,

$$\lim_{N\to\infty}\int_{et}^{Nt} \frac{\sin x}{x}\,dx$$

exists. Theorem 2B then shows that, as $N \to \infty$, the right side of (2) approaches a finite limit. But this would imply

$$\lim_{N\to\infty}\int_e^N \frac{g(x)}{x} < \infty,$$

which contradicts (1). This contradiction proves that g is not a Fourier transform.

B

5. Inversion of the Fourier transform

5A. In 3A we defined $\hat{f}$ for $f \in L^1$. We now ask—if we know that a function $\hat{f}$ is the Fourier transform of some $f \in L^1$, can we determine the function f from the values $\hat{f}(x)$ of $\hat{f}$? In other words, can we invert the Fourier transform? The answer, suitably interpreted, is 'yes'. *Formally*, the inversion is as follows:

If
$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt,$$
then
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} \hat{f}(x)\, dx. \tag{1}$$

However, there are Fourier transforms $\hat{f}$ for which the integral in (1) does not exist as a Lebesgue integral. (That is, $\hat{f}$ need not be in L^1.) For example, if
$$f(t) = e^{-t} \quad (t \geqslant 0),$$
$$= 0 \quad (t < 0),$$
then $\hat{f}(x) = \dfrac{1}{1 - ix}$.

In order for (1) to hold for a given value of t, special conditions must be imposed on f near t and a suitable interpretation must be given the integral in (1).

In this section and the next we shall make the above remarks precise. We begin with a lemma.

5B. Lemma. Let α be of bounded variation on $[0, \delta]$ for some $\delta > 0$. Then
$$\lim_{R\to\infty} \frac{1}{\pi}\int_0^\delta \alpha(t)\frac{\sin Rt}{t}\, dt = \tfrac{1}{2}\alpha(0^+).$$

Proof. Since any function of bounded variation can be expressed as the difference of two bounded non-decreasing functions, it is sufficient to prove the lemma in the case where α is non-decreasing and bounded on $[0, \delta]$.

Case 1. $\alpha(0^+) = 0$.

Given $\epsilon > 0$ we can find η with $0 < \eta < \delta$ such that $|\alpha(t)| \leqslant \epsilon$ for $0 < t < \eta$. By Theorem 2P there exists ξ in the interval $[0, \eta]$ such that
$$\int_0^\eta \alpha(t)\frac{\sin Rt}{t}\, dt = \alpha(\eta^-)\int_\xi^\eta \frac{\sin Rt}{t}\, dt + \alpha(0^+)\int_0^\xi \frac{\sin Rt}{t}\, dt.$$