CHAPTER 1

GENERAL PROPERTIES OF CONVEX SETS

1. Preliminaries and notation

All the numbers that are used in this book are real numbers. Complex numbers are never used at any stage.

The sets of points with which we shall be concerned will all be subsets of a real n-dimensional Euclidean space. Points will be denoted by lower case clarendon letters and sets by German capitals, except that we use the lower case clarendon letter of a point for the set consisting of that single point. The frontier of a set $\mathfrak{X}$ we denote by Fr $\mathfrak{X}$, its interior by $\mathfrak{X}^0$ and its closure by $\overline{\mathfrak{X}}$. In Chapters 6 and 7 we shall sometimes use capital letters to denote points. Numbers will be denoted by small Roman or Greek letters. n-Dimensional Euclidean space itself will be denoted by R^n. It is the class of all ordered sets of n real numbers $\mathbf{x}=(x_1, x_2, \ldots, x_n)$ made into a metric space by defining the distance between $\mathbf{x}$ and $\mathbf{y}$, where $\mathbf{y}$ is $(y_1, y_2, \ldots, y_n)$, to be

$$|\mathbf{x}-\mathbf{y}|=\left[\sum_{i=1}^{n}(x_i-y_i)^2\right]^{\frac{1}{2}}.$$

It is convenient to regard $\mathbf{x}$ as a vector and to define $\lambda\mathbf{x}$, $\mathbf{x}+\mathbf{y}$, $\mathbf{x}.\mathbf{y}$, $\mathbf{x}-\mathbf{y}$, by $\lambda\mathbf{x}=(\lambda x_1, \ldots, \lambda x_n)$, $\mathbf{x}+\mathbf{y}=(x_1+y_1, \ldots, x_n+y_n)$, $\mathbf{x}.\mathbf{y}=x_1y_1+\ldots+x_ny_n$, $\mathbf{x}-\mathbf{y}=\mathbf{x}+(-\mathbf{y})$ respectively where λ is a real number.

The origin in R^n is denoted by $\mathbf{O}$. The line joining the two distinct points $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{xy}$.

The symbol $\{\}$ will be used to indicate sets which satisfy conditions that will be stated explicitly inside the braces.

The empty set or the void set, regarded as a subset of R^n, will be denoted by ϕ.

The distance function has the following properties:

(i) $|\mathbf{x}-\mathbf{y}|\geqslant 0$ and $|\mathbf{x}-\mathbf{y}|=0$ if and only if $\mathbf{x}$ is $\mathbf{y}$.

(ii) $|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|\geqslant|\mathbf{x}-\mathbf{z}|$.

(iii) $|\mathbf{x}-\mathbf{y}| = |\mathbf{y}-\mathbf{x}|$.
(iv) $|\mathbf{x}-(\mathbf{y}+\mathbf{z})| = |(\mathbf{x}-\mathbf{y})-\mathbf{z}|$.
(v) $|\lambda\mathbf{x}-\lambda\mathbf{y}| = \lambda|\mathbf{x}-\mathbf{y}|$ if $\lambda \geqslant 0$.

The sphere whose centre is $\mathbf{x}$ and whose radius is r is denoted by $\mathfrak{S}(\mathbf{x}, r) = \{\mathbf{y} : |\mathbf{x}-\mathbf{y}| < r\}$. A subset $\mathfrak{X}$ of R^n is bounded if there exists a sphere $\mathfrak{S}(\mathbf{x}, r)$ such that $\mathfrak{X} \subset \mathfrak{S}(\mathbf{x}, r)$. An important property of such a set is that for any $r' > 0$ it contains at most a finite number of disjoint spheres of radius r'.

The distance of a point $\mathbf{x}$ from a set $\mathfrak{Y}$ is defined by

$$\rho(\mathbf{x}, \mathfrak{Y}) = \inf\{|\mathbf{x}-\mathbf{y}| : \mathbf{y} \in \mathfrak{Y}\}.$$

A set of the form $\mathfrak{U}(\mathfrak{Y}, r) = \{\mathbf{x} : \rho(\mathbf{x}, \mathfrak{Y}) < r\}$ where $r > 0$ is said to be a neighbourhood of $\mathfrak{Y}$.

We shall use the sign $+$ both for the addition of numbers and for the vector addition of sets (cf. Chapter 1, § 2, Chapter 5), but not for unions of sets. However, for differences of sets, the intersection-complement notation is cumbersome, and so we shall use the notation $\mathfrak{X} \dot{-} \mathfrak{Y}$ to indicate the set of points which belong to $\mathfrak{X}$ and not to $\mathfrak{Y}$.

The volume of a set in R^n is its n-dimensional Lebesgue measure. We denote the volume of the set $\mathfrak{X}$ by $V(\mathfrak{X})$ or by $V_n(\mathfrak{X})$ if we wish to emphasize the dimension of the space in which we are working. The volume of a set in R^2 is usually referred to as its area and the volume of a set in R^1 as its length. We shall later define the area of a set in a different manner, but there should be no confusion between the two quite distinct usages of the word. The properties of Lebesgue measure that we use may be found in any standard text-book in the subject. The following properties are particularly important:

(i) $V_n(\mathfrak{X}) > 0$ if $\mathfrak{X}$ contains a sphere in R^n. $V_n(\mathfrak{X}) = 0$ if $\mathfrak{X}$ is contained in an $(n-1)$-dimensional linear manifold of R^n, i.e. the subset of points $\mathbf{x} = (x_1, \ldots, x_n)$ which satisfy an equation of the form

$$a_1 x_1 + \ldots + a_n x_n = b,$$

where not all the a_i are zero. Such a set is also referred to as a *hyperplane*.

(ii) If $\mathfrak{X} \supset \mathfrak{Y}$ then $V(\mathfrak{X}) \geqslant V(\mathfrak{Y})$.

(iii) Let $\mathbf{u}$ be a fixed unit vector and $\mathfrak{P}(\lambda)$ be the hyperplane $\mathbf{x}.\mathbf{u}=\lambda$; then

$$V(\mathfrak{X})=\int_{-\infty}^{+\infty} V_{n-1}(\mathfrak{P}(\lambda)\cap\mathfrak{X})\,d\lambda,$$

where the integral is a Lebesgue integral.

(iv) If $\mathfrak{X}_1$ is similar to $\mathfrak{X}_2$ and the ratio of similitude is $\mu:1$ then

$$V(\mathfrak{X}_1)=\mu^n V(\mathfrak{X}_2).$$

We shall consistently ignore questions of measurability. The sets which are discussed will, in fact, all be measurable but no proofs will be given.

It is assumed that the reader is familiar with the simpler concepts of topology.

The expression $f(\mathbf{x})=a_1\mathbf{x}_1+\ldots+a_n\mathbf{x}_n-b$ where not all the numbers a_i are zero, defines a hyperplane $\mathfrak{P}=\{\mathbf{x}:f(\mathbf{x})=0\}$. The two sets $\{\mathbf{x}:f(\mathbf{x})>0\}$ and $\{\mathbf{x}:f(\mathbf{x})<0\}$ are called open half-spaces and are said to be bounded by $\mathfrak{P}$. Similarly, the sets $\{\mathbf{x}:f(\mathbf{x})\geqslant 0\}$ and $\{\mathbf{x}:f(\mathbf{x})\leqslant 0\}$ are closed half-spaces and are also said to be bounded by $\mathfrak{P}$. If two subsets of R^n, say $\mathfrak{X}_1$ and $\mathfrak{X}_2$, are such that one of the open half-spaces bounded by $\mathfrak{P}$ contains $\mathfrak{X}_1$ and the other open half-space contains $\mathfrak{X}_2$, we say that $\mathfrak{P}$ *separates* the sets $\mathfrak{X}_1$ and $\mathfrak{X}_2$ *strictly*. If one of the closed half-spaces contains $\mathfrak{X}_1$ and the other closed half-space contains $\mathfrak{X}_2$, then we say that $\mathfrak{P}$ *separates* $\mathfrak{X}_1$ and $\mathfrak{X}_2$.

2. The definition of convexity and its relation to affine transformations

DEFINITION. *A set of points $\mathfrak{X}$ is said to be convex if whenever two points $\mathbf{x}_1, \mathbf{x}_2$ belong to $\mathfrak{X}$ all the points of the form*

$$\lambda\mathbf{x}_1+\mu\mathbf{x}_2 \tag{1·2·1}$$

where $\lambda\geqslant 0$, $\mu\geqslant 0$, $\lambda+\mu=1$ also belong to $\mathfrak{X}$.

Or, to put the definition in a more geometrical language: $\mathfrak{X}$ is convex if whenever it contains two points it also contains the segment joining these two points. This may be put into symbols by writing $\mathfrak{H}(\mathbf{x}_1,\mathbf{x}_2)$ for the closed segment joining the points $\mathbf{x}_1, \mathbf{x}_2$. Then the definition is: $\mathfrak{X}$ is convex if and only if $\mathbf{x}_1\in\mathfrak{X}$, $\mathbf{x}_2\in\mathfrak{X}$ imply $\mathfrak{H}(\mathbf{x}_1,\mathbf{x}_2)\subset\mathfrak{X}$.

Examples of convex sets are: the empty set, a set consisting of a single point, a segment, a sphere, and a regular polygon. To avoid the necessity of considering tiresome trivial but exceptional cases, we make the convention that the empty set is not a convex set, and we shall use the phrase 'convex set' always to mean 'non-empty convex set'.

It should be noted that the definition does not depend upon the dimension of the Euclidean space concerned and that it is possible to use the same definition in much more general spaces. If $\mathfrak{X}_1$ is a convex subset of R^n and $\mathfrak{X}_2$ is a convex subset of R^m then the Cartesian product of $\mathfrak{X}_1, \mathfrak{X}_2$ denoted by $\mathfrak{X}_1 \times \mathfrak{X}_2$ and defined by

$$\mathfrak{X}_1 \times \mathfrak{X}_2 = \{(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) : (x_1, \ldots, x_n) \in \mathfrak{X}_1, (x_{n+1}, \ldots, x_{n+m}) \in \mathfrak{X}_2\}$$

is a convex subset of $\mathsf{R}^n \times \mathsf{R}^m$. We identify this last space with R^{n+m}.

The definition can be put in many different forms; some of these will be discussed later. It is often convenient to use a somewhat more general form of the convexity conditions which may be stated as follows. If $\mathfrak{X}$ is a convex set and if $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are k points of $\mathfrak{X}$, then every point $\mathbf{x}$ of the form

$$\mathbf{x} = \lambda_1 \mathbf{x}_1 + \ldots + \lambda_k \mathbf{x}_k \quad (\lambda_i \geq 0, \lambda_1 + \ldots + \lambda_k = 1) \tag{1·2·2}$$

also belongs to $\mathfrak{X}$. For $k = 2$ this is just the definition that $\mathfrak{X}$ is convex. To prove the truth of (1·2·2) assume inductively that it is true for $k = m$ and consider a point $\mathbf{x}$ of the form

$$\mathbf{x} = \lambda_1 \mathbf{x}_1 + \ldots + \lambda_m \mathbf{x}_m + \lambda_{m+1} \mathbf{x}_{m+1} \quad (\lambda_i \geq 0, \lambda_1 + \ldots + \lambda_{m+1} = 1). \tag{1·2·3}$$

If $\lambda_{m+1} = 1$ then $\mathbf{x} = \mathbf{x}_{m+1}$ which belongs to $\mathfrak{X}$ and there is nothing further to prove. Suppose then that $\lambda_{m+1} < 1$. In this case

$$\lambda_1 + \ldots + \lambda_m = 1 - \lambda_{m+1} > 0,$$

and we may put $\mathbf{x}$ in the form

$$\mathbf{x} = (\lambda_1 + \ldots + \lambda_m)\left(\frac{\lambda_1}{\lambda_1 + \ldots + \lambda_m}\mathbf{x}_1 + \ldots + \frac{\lambda_m}{\lambda_1 + \ldots + \lambda_m}\mathbf{x}_m\right) + \lambda_{m+1}\mathbf{x}_{m+1}. \tag{1·2·4}$$

By the inductive hypothesis the point

$$\mathbf{z} = \frac{\lambda_1}{\lambda_1 + \ldots + \lambda_m}\mathbf{x}_1 + \ldots + \frac{\lambda_m}{\lambda_1 + \ldots + \lambda_m}\mathbf{x}_m$$

belongs to $\mathfrak{X}$. Since $\mathfrak{X}$ is convex and contains both $\mathbf{z}$ and $\mathbf{x}_{m+1}$, it follows from (1·2·4) that it contains $\mathbf{x}$. Thus the inductive hypothesis is proved when $k = m+1$. Hence it is true for all k.

In many ways it is more correct to regard convexity as a property of affine geometry rather than as a property of Euclidean geometry. We do not adopt this point of view here, since there

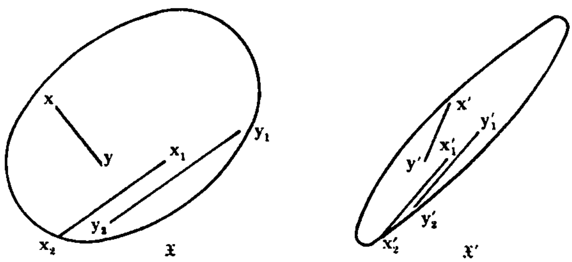

Fig. 1

are some properties of convex sets which are not invariant under affine transformations. But it is important to see what effect an affine transformation will have on the convex subsets of R^n.

A point transformation from $\mathbf{x}$ to $\mathbf{x}'$ is said to be affine if there exists a relation of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}, \tag{1·2·5}$$

where A is a non-singular $n \times n$ real matrix, and $\mathbf{b}$ is a vector of R^n.

Now if $\mathbf{z} = \lambda\mathbf{x} + \mu\mathbf{y}$, where $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu = 1$, and if $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are transformed by (1·2·5) into $\mathbf{x}', \mathbf{y}', \mathbf{z}'$ respectively, then

$$\mathbf{z}' = \lambda\mathbf{x}' + \mu\mathbf{y}'.$$

Thus the segment $\mathfrak{H}(\mathbf{x}, \mathbf{y})$ is transformed into the segment $\mathfrak{H}(\mathbf{x}', \mathbf{y}')$ and a convex set $\mathfrak{X}$ is transformed into another convex set $\mathfrak{X}'$ (see fig. 1).

If
$$\mathbf{x}_1 - \mathbf{x}_2 = \lambda(\mathbf{y}_1 - \mathbf{y}_2),$$
and if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ are transformed respectively into $\mathbf{x}_1', \mathbf{x}_2', \mathbf{y}_1', \mathbf{y}_2'$, then
$$\mathbf{x}_1' - \mathbf{x}_2' = \lambda(\mathbf{y}_1' - \mathbf{y}_2').$$
Thus parallel lines are transformed into parallel lines and the ratio of the lengths of segments lying in parallel lines is not altered by the transformation. A similar statement is true of areas lying in parallel planes, etc. The actual change in the length of the segment depends upon the transformation, and in general varies as the inclination of the segment to the axes of coordinates is varied.

The change in the n-dimensional volume $V(\mathfrak{X})$ of a set $\mathfrak{X}$ is given by
$$V(\mathfrak{X}') = |\det \mathrm{A}|\, V(\mathfrak{X}),$$
where det A is the determinant of the matrix A.

It is sometimes convenient to reduce a particular problem to a simpler form by using an affine transformation. A useful property in this connexion is the following. Suppose that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ and $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ are two sets of n vectors and that each set is linearly independent; then there exists an affine transformation which sends $\mathbf{x}^{(i)}$ into $\mathbf{y}^{(i)}$, $i = 1, 2, \ldots, n$, and $\mathbf{O}$ into $\mathbf{O}$. To establish the correctness of this statement let L be the $n \times n$ matrix whose columns are $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ and M be the $n \times n$ matrix whose columns are $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$. Since L is non-singular we can define a matrix
$$\mathrm{A} = \mathrm{ML}^{-1}.$$
Since M is non-singular, it follows that A is a real non-singular matrix. It is the matrix of the required affine transformation of the form (1·2·5) with $\mathbf{b} = \mathbf{O}$.

If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n+1)}$ are the vertices of a non-degenerate simplex in R^n and if $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n+1)}$ are those of a regular non-degenerate simplex, then there exists an affine transformation which transforms $\mathbf{x}^{(i)}$ into $\mathbf{y}^{(i)}$, $i = 1, 2, \ldots, n+1$. For the two sets of n vectors $\mathbf{x}^{(i)} - \mathbf{x}^{(1)}$ and $\mathbf{y}^{(i)} - \mathbf{y}^{(1)}$, $i = 2, \ldots, n+1$, are each linearly independent. Thus there exists an affine transformation of the form (1·2·5) which transforms $\mathbf{x}^{(i)} - \mathbf{x}^{(1)}$ into $\mathbf{y}^{(i)} - \mathbf{y}^{(1)}$. Let the matrix of this transformation be A. Then
$$\mathbf{y}^{(i)} - \mathbf{y}^{(1)} = \mathrm{A}(\mathbf{x}^{(i)} - \mathbf{x}^{(1)}) \quad (2 \leqslant i \leqslant n+1). \tag{1·2·6}$$

Define the vector $\mathbf{b}$ by

$$\mathbf{b} = -\mathrm{A}\mathbf{x}^{(1)} + \mathbf{y}^{(1)}. \qquad (1\cdot2\cdot7)$$

(1·2·6) and (1·2·7) together imply that

$$\mathbf{y}^{(i)} = \mathrm{A}\mathbf{x}^{(i)} + \mathbf{b} \quad (1 \leqslant i \leqslant n+1).$$

This is the required transformation of the form (1·2·5).

Degenerate affine transformations which arise when the matrix A is singular can also be used. In particular if A is an $m \times n$ matrix with $m < n$, and if A is of rank m then the transformation $\mathbf{x}' = \mathrm{A}.\mathbf{x}$, where $\mathbf{x} \in \mathsf{R}^n$ and $\mathbf{x}' \in \mathsf{R}^m$ defines an affine transformation of R^n onto R^m. This transformation also preserves convexity.

If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are any two sets in R^n, we can define a third set $\mathfrak{X}_3$, which we denote by $\lambda\mathfrak{X}_1 + \mu\mathfrak{X}_2$, by the relation

$$\mathfrak{X}_3 = \{\mathbf{x} : \mathbf{x} = \lambda\mathbf{x}_1 + \mu\mathbf{x}_2, \mathbf{x}_1 \in \mathfrak{X}_1, \mathbf{x}_2 \in \mathfrak{X}_2\}.$$

If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are convex then $\mathfrak{X}_3$ is convex. This fact may either be proved directly or can be obtained by using a degenerate affine transformation as follows. $\mathfrak{X}_1 \times \mathfrak{X}_2$ is a convex subset of R^{2n} and the $n \times 2n$ matrix A whose elements a_{ij} are defined by $a_{ii} = \lambda$, $a_{i,n+i} = \mu$, $a_{ij} = 0, j \neq i, n+i$, where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant 2n$ gives rise to an affine mapping of R^{2n} onto R^n and of $\mathfrak{X}_1 \times \mathfrak{X}_2$ onto $\mathfrak{X}_3$. Thus $\mathfrak{X}_3$ is convex.

EXERCISES 1·2

1. $\mathfrak{X}$ is a convex set in R^n and $\mathbf{y}$ does not belong to $\mathfrak{X}$. $\mathfrak{H}(\mathbf{y}, \mathfrak{X})$ is the point-set union of all the segments one of whose end-points is $\mathbf{y}$ and the other is a point of $\mathfrak{X}$. Show that $\mathfrak{H}(\mathbf{y}, \mathfrak{X})$ is convex.

2. $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are two disjoint convex sets in R^n and $\mathbf{y}$ does not belong to $\mathfrak{X}_1 \cup \mathfrak{X}_2$. With the notation of Exercise 1·2(1) show that either $\mathfrak{H}(\mathbf{y}, \mathfrak{X}_1) \cap \mathfrak{X}_2 = \phi$ or $\mathfrak{H}(\mathbf{y}, \mathfrak{X}_2) \cap \mathfrak{X}_1 = \phi$.

3. In R^n, n hyperplanes pass through the origin and the origin is their only common point. Show that there is an affine transformation which transforms these hyperplanes into hyperplanes that are perpendicular to one another.

3. Intersections, closures and interiors of convex sets

In dealing with point-sets of a general nature it is usually essential to make some assertion as to whether they are open or closed or neither, and the properties of these three classes of sets are widely different. In the case of convex sets, although a given convex set may be neither open nor closed, yet, since both its closure and its interior (if this is non-empty) are also convex, we can infer the properties of general convex sets from those of closed convex sets or from those of open convex sets. This is usually so simple a matter that it is customary to develop the detailed theory of convex sets for closed sets only. There are some problems which cannot be simplified in this fashion, and for this reason we do not restrict our attention to closed sets until we reach Chapter 4.

In the present section we establish the convexity of the closure of a convex set and the convexity of the non-empty interior of a convex set.

THEOREM 1. *If* $\mathfrak{I}$ *is an index set, if* $\mathfrak{X}_i$ *for each* $i \in \mathfrak{I}$ *is convex and if the intersection* $\bigcap_{i \in \mathfrak{I}} \mathfrak{X}_i$ *is non-empty, then this intersection is convex.*

If $\mathbf{x}$ and $\mathbf{y}$ belong to $\bigcap_{i \in \mathfrak{I}} \mathfrak{X}_i$, they also belong to $\mathfrak{X}_i$. Since $\mathfrak{X}_i$ is convex $\mathfrak{H}(\mathbf{x}, \mathbf{y}) \subset \mathfrak{X}_i$, and since this is true for every i, $i \in \mathfrak{I}$, $\mathfrak{H}(\mathbf{x}, \mathbf{y}) \subset \bigcap_{i \in \mathfrak{I}} \mathfrak{X}_i$. Thus $\bigcap_{i \in \mathfrak{I}} \mathfrak{X}_i$ is convex.

COROLLARY. *If* $(\mathfrak{X}_i)$ *is a sequence of convex sets, then*

$$\liminf \mathfrak{X}_i = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} \mathfrak{X}_i$$

is convex or is empty.

For by the theorem each set $\bigcap_{i=k}^{\infty} \mathfrak{X}_i$ is convex or empty. If each of these sets is empty then so is $\liminf \mathfrak{X}_i$. If one of them is not empty then $\liminf \mathfrak{X}_i$ is the union of an increasing sequence of convex sets and as such is easily seen to be convex.

THEOREM 2. *If* $\mathfrak{X}$ *is a convex set and* δ *a positive number, then the set* $\mathfrak{U}(\mathfrak{X}, \delta)$ *is also convex.*

Since $\mathfrak{U}(\mathfrak{X}, \delta)$ contains $\mathfrak{X}$ it is non-empty. Let $\mathbf{y}$ and $\mathbf{y}'$ be two points of $\mathfrak{U}(\mathfrak{X}, \delta)$ (see fig. 2). Then since

$$\mathfrak{U}(\mathfrak{X}, \delta) = \{\mathbf{z} : \rho(\mathbf{z}, \mathfrak{X}) < \delta\},$$

there are at least two points $\mathbf{x}$ and $\mathbf{x}'$ of $\mathfrak{X}$ such that

$$|\mathbf{x}-\mathbf{y}|<\delta, \quad |\mathbf{x}'-\mathbf{y}'|<\delta.$$

Let $\mathbf{y}''$ be a point of $\mathfrak{H}(\mathbf{y},\mathbf{y}')$, say

$$\mathbf{y}''=\lambda\mathbf{y}+\mu\mathbf{y}', \tag{1·3·1}$$

where $\lambda \geqslant 0$, $\mu \geqslant 0$, $\lambda+\mu=1$, and define $\mathbf{x}''$ to be the point

$$\mathbf{x}''=\lambda\mathbf{x}+\mu\mathbf{x}' \tag{1·3·2}$$

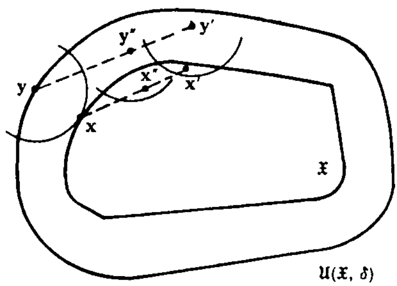

Fig. 2

with the same values of λ and μ as in (1·3·1). Then $\mathbf{x}\in\mathfrak{X}$ (since $\mathfrak{X}$ is convex) and

$$\begin{aligned}|\mathbf{y}''-\mathbf{x}''| &\leqslant |\lambda\mathbf{x}-\lambda\mathbf{y}|+|\mu\mathbf{x}'-\mu\mathbf{y}'|\\ &=\lambda|\mathbf{x}-\mathbf{y}|+\mu|\mathbf{x}'-\mathbf{y}'|\\ &<\delta.\end{aligned}$$

Hence $\mathbf{y}''\in\mathfrak{U}(\mathfrak{X},\delta)$, and it follows that $\mathfrak{U}(\mathfrak{X},\delta)$ is convex.

COROLLARY. *If $\mathfrak{X}$ is convex so is $\overline{\mathfrak{X}}$, the closure of $\mathfrak{X}$.*

For if $\{\delta_i\}$ is a sequence of positive numbers decreasing to zero, then

$$\overline{\mathfrak{X}}=\bigcap_{i=1}^{\infty}\mathfrak{U}(\mathfrak{X},\delta_i),$$

and the result follows from Theorems 1 and 2.

THEOREM 3. *Let $\mathfrak{X}$ be a convex set with a non-void interior $\mathfrak{X}^0$, and let $\mathbf{x}_1$, $\mathbf{x}_2$ be two points of $\mathfrak{X}$, of which $\mathbf{x}_2$ belongs to $\mathfrak{X}^0$. Then every point of $\mathfrak{H}(\mathbf{x}_1,\mathbf{x}_2)$, except possibly $\mathbf{x}_1$, is an interior point of $\mathfrak{X}$.*

We know that every point of $\mathfrak{H}(\mathbf{x}_1, \mathbf{x}_2)$ belongs to $\mathfrak{X}$, since $\mathfrak{X}$ is convex. We have to show that each such point, apart from $\mathbf{x}_1$, is also an interior point of $\mathfrak{X}$.

Since $\mathbf{x}_2$ is an interior point of $\mathfrak{X}$ there is a positive number δ such that

$$\mathfrak{S}(\mathbf{x}_2, \delta) \subset \mathfrak{X}. \tag{1·3·3}$$

Let $\mathbf{y}$ be a point of $\mathfrak{H}(\mathbf{x}_1, \mathbf{x}_2)$, not identical with $\mathbf{x}_1$, say

$$\mathbf{y} = \lambda \mathbf{x}_1 + \mu \mathbf{x}_2, \tag{1·3·4}$$

where $\lambda \geqslant 0$, $\mu > 0$, $\lambda + \mu = 1$. Let $\mathbf{z}$ be a point of $\mathfrak{S}(\mathbf{y}, \mu\delta)$ (see fig. 3). Then $|\mathbf{z} - \mathbf{y}| < \mu\delta$, or, from (1·3·4),

$$|\mathbf{z} - (\lambda \mathbf{x}_1 + \mu \mathbf{x}_2)| < \mu\delta.$$

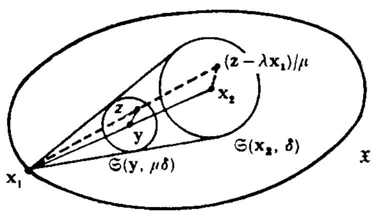

Fig. 3

Since $\mu > 0$, this inequality implies

$$|(\mathbf{z} - \lambda \mathbf{x}_1)/\mu - \mathbf{x}_2| < \delta. \tag{1·3·5}$$

From (1·3·3) and (1·3·5) it follows that $(\mathbf{z} - \lambda \mathbf{x}_1)/\mu$ belongs to $\mathfrak{X}$. But since $\mathfrak{X}$ is convex and since $\mathbf{z}$ can be written in the form

$$\mathbf{z} = \mu[(\mathbf{z} - \lambda \mathbf{x}_1)/\mu] + \lambda \mathbf{x}_1,$$

it follows that $\mathbf{z} \in \mathfrak{X}$.

Thus $\mathfrak{S}(\mathbf{y}, \mu\delta) \subset \mathfrak{X}$, and hence $\mathbf{y}$ is an interior point of $\mathfrak{X}$.

COROLLARY 1. *If $\mathfrak{X}$ is convex then $\mathfrak{X}^0$, the interior of $\mathfrak{X}$, is either convex or empty.*

COROLLARY 2. *The theorem is still true if instead of being given that $\mathbf{x}_1 \in \mathfrak{X}$ we are given only that $\mathbf{x}_1 \in \overline{\mathfrak{X}}$.*

Since $\mathbf{x}_2 \in \mathfrak{X}^0$ there exists $\delta > 0$ such that $\mathfrak{S}(\mathbf{x}_2, \delta) \subset \mathfrak{X}^0$. Consider $\mathbf{y} \in \mathfrak{H}(\mathbf{x}_1, \mathbf{x}_2)$ and suppose that $\mathbf{y} \neq \mathbf{x}_1, \mathbf{x}_2$. Let $\mathbf{z}_1$ be a point of $\mathfrak{X}$ that satisfies

$$|\mathbf{z}_1 - \mathbf{x}_1| < \delta |\mathbf{x}_1 - \mathbf{y}| / |\mathbf{x}_2 - \mathbf{y}|.$$