CAMBRIDGE TRACTS IN MATHEMATICS

65. Normal topological spaces

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CORRIGENDA

Page 66 (lines 13 and 16): FOR $Y-f(x)$ READ $Z-f(x)$
Page 66 (line 14): FOR image of X READ image of βX
Page 67 (line 17): FOR rings of sets READ strong delta normal bases
Page 68 (line 21): DELETE if and
Page 68 (line 23): FOR $\mathrm{cl}_{vX} S$ READ $\mathrm{cl}_{vY} S$
Page 147 (4th line from foot): FOR For every f is READ For every f in

Introduction

The origins of general set theoretical topology go back to the works of Alexandroff, Fréchet, Hausdorff, and Urysohn, to mention just a few of the more important names. It interacts with various other branches of mathematics such as functional analysis, modern algebra, the theory of partial differential equations, algebraic topology, differential topology and differential geometry. It was influenced strongly by algebraic geometry, functional analysis, and the theories of real and complex variables (the last as commenced by the early works of Weierstrass). With these various elements interacting, new advances in general topology have not in general been under the influence of the more powerful developments in other areas of mathematics. Hence one of the objectives of general topology has been to find and develop a unified system and unified approach to this area. Terms and ideas which appear dissimilar at first should be analyzed and new terms and ideas invented to encompass these. Considerable progress has been made by the Soviet mathematicians in this area. Behind much of this work is found the notions (and variations of these notions) of locally finite covers, star-finite refinements, normal sequences of covers as well as alterations of the notion of pseudometrics and concepts from dimension theory.

Completely regular spaces have been shown to be natural generalizations of metric spaces since they are precisely the ones that have Hausdorff compactifications, they are the spaces that are embeddable in cubes, and they are the spaces which are associated with uniformities where the metric space notions of completeness and Cauchy sequences have their most general setting. (Hence the class of uniform spaces are important spaces for consideration.) However a slightly stronger condition than completely regular has been found to be more influential in the development of a unifying theory (however, by no means completely satisfactory, yet more satisfactory than completely regular spaces). This is the notion of a normal space. Even though the class of normal spaces is neither hereditary nor productive (as the completely regular spaces are), they do have significant properties. They are, for example, characterized among the spaces that are uniformizable as those spaces for which

(1) every closed subspace is C-embedded, and

(2) every open point-finite cover has associated with it an open cover with the property that the closures of its members refines the original cover.

This gives information as to when a cover of a normal space is a normal cover, one of the important notions mentioned above. Hence it is worthwhile to investigate characterizations of normal spaces in the light of recent developments and to utilize these notions to arrive at again new concepts useful in obtaining a unified theory.

Paracompact spaces have also had their impact on general topology. It is in the theory of these spaces that there is couched the solution to the metrization problem. Again the notion lends itself readily to the notions of normal covers. For purposes of dimension theory there is some doubt whether general paracompact spaces or general normal spaces are adequate for extending the results of the already well-established theory for metric spaces. As mentioned above new ideas have evolved which along with paracompactness and normality have assisted in giving meaningful results. For example such a notion would be that of a paracompact M-space (this last having been introduced by K. Morita). Every metric space and every compact space are paracompact M-spaces and the countable product of paracompact M-spaces is of the same type.

Thus we arrive at some of the reasons for the present text. It is clear that today in the field of general set theoretical topology a text is needed to discuss material beyond the usual basic general topology course. However we feel that this 'second text' should serve a number of roles which, as they turn out, prove to be closely related, one to the others. It is desirable to present to the student recent developments in general topology. It is desirable to endow him with the impetus, and the inspiration to carry on mathematical research. For this it is hoped that the material presently at hand will give the necessary new ideas, the necessary techniques, and the necessary goals for achieving meaningful research in set theoretical topology. But what precisely does this phrase, 'meaningful research in set theoretical topology', mean? We have pointed out some of the areas that have influenced it and those that it has inspired. We have also discussed the need for defining a methodology and for implementing new ideas to increase the advances in this area as well as to further inspire other areas.

Here then, we discuss in great detail the notion of normal cover and the related uniform concepts. We look at many characterizations of normal spaces and their interrelations with paracompact spaces and the various ways of embedding subspaces. We also look at the new concepts that have made such a recent impact (such as M-spaces) and consider their relationship to the above concepts.

With this material then and with the many research problems suggested by it, we hope to assist in this last aim. A prerequisite for the text is of course a basic general topology course which may have hinted at the notions of paracompactness and uniform spaces (but it is not absolutely necessary that they did). We utilize much of the theory of *Rings of Continuous Functions* as developed in [XI]. However, the necessary tools for this work are developed here, leaving for additional leisure reading only the notions whose techniques are not pertinent to the ensuing discussion. Bringing in this area (*Rings of Continuous Functions*) also assists in the interaction of general topology with other branches of mathematics.

As to the structure of the text, first of all, we have made an attempt to gather many of the characterizations of both a normal space and a collectionwise normal space into one text. Around this basic theme, the following course has been developed.

Chapter I contains basic preliminary results. All of these results (along with some in Chapter II) except those on uniform spaces, should be familiar to anyone with a basic course in topology. In Chapter II, the structural relationship between normal spaces and the real-valued continuous functions defined on them, is considered. Here also are stated and proved the known characterizations of normality in terms of C-embedding and C^*-embedding. Characterizations of normal spaces and of collectionwise normal spaces in terms of various open covers are given in Chapter III. Then a list of equivalent conditions for a topological space to be normal are stated and, although many of these conditions have appeared in one place or another in the literature, there has been no attempt to organize them and to assimilate the ideas which they encompass. These then are used as a constructing frame on which is built the entire course. To assist in this task, many theorems about refinement of covers that are part of the folklore of this subject but never seem to be published are also shown here. Then a proof of Katětov's theorem characterizing collectionwise normal spaces is given.

In Chapter IV the extension of pseudometrics is discussed. Using various classes of pseudometrics, normal spaces and collectionwise normal spaces are characterized. It is shown that, roughly speaking, extending continuous pseudometries is to collectionwise normal spaces as extending real-valued continuous functions is to normal spaces.

In Chapter V a study is made of extending uniformities. These concepts are then used to give additional characterizations of normal and collectionwise normal spaces. Also in this chapter we make a point to emphasize some additional applications of the new results brought forth in the text.

In the Appendix, we present no proofs of theorems but only a survey of some ideas presently under consideration in this fast developing field of general topology. We hope that this will give the reader some idea of the many problems presently being researched and at the same time encourage him to search the literature for the proofs and additional ideas. In a book such as this, it is impossible to include many of the ideas that seem to be worthwhile. We have tried to select some which appear pertinent to the results in the text.

It is difficult to recognize all those who have contributed, in one way or another, to the development of this book. We have taught the material to several classes of graduate students and to all of them we owe a debt of thanks for their patience, their corrections and their suggestions. Their number prohibits us from naming them all. On the other hand, we must name and thank the following students who worked more closely with us – Jogendra Kumar Kohli, Arvind Kumar Misra, Greg Naber, Linn Imler Sennott, and Maury Weir. They read parts of the manuscript at various stages and some contributed either new results or assisted in the exposition of some of the areas. We would also like to thank Professors Robert Heath, Jun-iti Nagata and Akihiro Okuyama for their many valuable comments, suggestions, and assistance in the preparation of Section 22 in the Appendix. Professor Mary Ellen Rudin was very kind to permit us to use a pre-print of her paper (see [333] and [417]) where she solves the long unsolved and difficult Dowker problem. Eddie Lamarr Buckner and Janice Frederick did a wonderful job in typing much of the manuscript. Professor C. T. C. Wall was very helpful with the editing. We wish to thank our respective departments of mathematics for providing research support for the completion of this project. Finally the second named author must thank the Mathematics Department at Carnegie-Mellon University for providing a two year visiting appointment which enabled the commencement of this work.

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This book is dedicated to
Elena Chimenti, Giuseppe Alò
Serafina Torchia
Melanie Jeanne, Kenneth Mark, Lili Rosanne
Michelina Luisa
and to
Flora and Maurice Shapiro
the many abstractions I have known
the memory of the Kent State massacre