I. *Preliminaries*

1 Notation and terminology

To establish a common foundation on which to build it is necessary to make a few comments about terminology and notation. This section is devoted to this and also to the recollecting (for some) of concepts from set theoretical topology. A quick perusal by the informed, then, should be useful.

The set of real numbers is denoted by **R**. Its subset of non-negative real numbers is denoted by $\mathbf{R}^+$, whereas its subset of positive integers is denoted by **N**. The set of real numbers x for which $0 \leqslant x \leqslant 1$ is denoted by E. These are the most commonly cited subsets of **R** used in this book. From time to time, other subsets are considered. However they will be defined when needed.

If X is a set, 2^X denotes the collection of all subsets of X and it is called the *power set of* X. A distinction is made between a collection $\mathscr{U}$ of subsets of a set X and a family $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ of subsets of a set X as follows. By a *family* $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ of subsets of X is meant that there exists a set I, called the *index set* of the family, and a function f mapping I into 2^X such that $f(\alpha) = U_\alpha$. Loosely speaking, a family of subsets is a collection of subsets in which each member has been labeled.

In the sequel definitions are given in terms of either families or collections and it is left to the reader to formulate the implicit definition using the other concept. In the text, free use is made of whichever concept seems the most natural.

Let $\mathscr{T}$ denote the collection of all open subsets of a topological space X. When it is desired to call particular attention to the topology $\mathscr{T}$ of X, or when the underlying point-set is to be provided with more than one topology, reference shall be made to X as '*the topological space* $(X, \mathscr{T})$'.

The concepts of a subspace and relative topologies (as in [XVIII]) are concerned with the reaction of a collection of subsets of a topological space on a distinguished subset. To be able to use the idea more fully the following definitions are made.

1.1 DEFINITION If A is a subset of the set X, and if $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ is a family of subsets of X, then by the *trace of* $\mathscr{U}$ on A is meant the family $(U_\alpha \cap A)_{\alpha \in I}$. The trace of $\mathscr{U}$ on A is denoted by $\mathscr{U}|A$. If $\mathscr{S}$ is a collection of subsets of X, then by the trace of $\mathscr{S}$ on A is meant the collection $\{S \cap A: S \in \mathscr{S}\}$ and it also is denoted by $\mathscr{S}|A$. Conversely if $\mathscr{S}$ is a collection of subsets of A then $\mathscr{U}$ is an extension of $\mathscr{S}$ if for each $S \in \mathscr{S}$ there is a $U \in \mathscr{U}$ such that $U \cap A = S$. (Thus the topology $\mathscr{T}$ on a set X is an extension of the relative topology on any of its subsets, whereas the relative topology on a subset A of X is just the trace of $\mathscr{T}$ on A.)

The *closure* of a subset A of X will be denoted by $\operatorname{cl} A$, or, when there is possibility of confusion, by $\operatorname{cl}_X A$. Analogously, the *interior* of a subset A of X will be denoted by $\operatorname{int} A$ or $\operatorname{int}_X A$. A subset G is *regular open* if $G = \operatorname{int} \operatorname{cl} G$ and a set F is *regular closed* if $F = \operatorname{cl} \operatorname{int} F$. The *complement* of a subset B of X will be denoted by $X - B$. If A and B are contained in X, then by $A - B$ is meant $\{x \in X: x \in A \text{ and } x \notin B\}$. The cardinal of a set X is denoted by $|X|$. The collection of all finite subsets of the set X will be denoted by $[X]$.

Often for collections of subsets of a set X, it is necessary to look at intersections of finite subcollections. More precisely, if $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ is a family of subsets of a set X, then by $\mathscr{U}^F$ is meant the family of subsets of X obtained by taking for each $J \in [I]$, the set $\cap \{U_\alpha: \alpha \in J\}$.

Many times in general topology a particular property which is defined on the entire space will have worthwhile implications when it is localized to certain open bases for points of the space. For example such has been the case with compactness and connectedness, being utilized in the motivation of local compactness and local connectedness. Dieudonné (see [95]) was the first to significantly localize the notion of compactness in yet another way called paracompactness. To pave the way for this definition it is necessary to look at properties of open covers of the whole space which also are local assertions. It is interesting to see that these notions which lead to Dieudonné's form of 'compactness' also lead to several characterizations of normality.

1.2 DEFINITION Let x be an element in the topological space X, and let $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ be a family of subsets of X. The family $(U_\alpha)_{\alpha \in I}$ is *point-finite at* x if there exists a finite subset K of I such that $x \notin U_\alpha$ for every $\alpha \notin K$. The family $(U_\alpha)_{\alpha \in I}$ is *locally finite at* x if there exist a neighborhood G of x and a finite subset J of I such that $G \cap U_\alpha = \varnothing$ for every $\alpha \notin J$. The family $(U_\alpha)_{\alpha \in I}$ is *discrete at* x if there exist a neighborhood H of x and a subset K of I with $|K| \leqslant 1$ such that $H \cap U_\alpha = \varnothing$ for every $\alpha \notin K$. The family $(U_\alpha)_{\alpha \in I}$ is *point-finite* (respectively, *locally finite*, respectively, *discrete*) if $(U_\alpha)_{\alpha \in I}$ is point-finite (respectively, locally finite, respectively, discrete) at every point $x \in X$. It will be said that $\mathscr{U}$ is *σ-locally finite* (respectively, *σ-point-finite*, respectively, *σ-discrete*) if $\mathscr{U}$ is a union of a countable number of families each of which is locally finite (respectively, point-finite, respectively, discrete). It is *closure preserving* if for every subset J of I

$$\operatorname{cl}\Big(\bigcup_{\alpha \in J} U_\alpha\Big) = \bigcup_{\alpha \in J} \operatorname{cl} U_\alpha.$$

A set G in a topological space $(X, \mathscr{T})$ is called a G_δ*-set* if G can be written as a countable intersection of open sets (that is, if there exists a sequence $(G_n)_{n \in \mathbf{N}} \subset \mathscr{T}$ such that $G = \bigcap_{n \in \mathbf{N}} G_n$). The set is called an F_σ*-set* if it can be written as a countable union of closed sets.

These concepts are applied as qualifiers to the concept of an open cover which is next to be defined.

1.3 DEFINITION Let $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ be a family of subsets of a set X, and let γ be a cardinal number. The family $\mathscr{U}$ is a *cover* of X in case $I \neq \varnothing$ and $X = \bigcup_{\alpha \in I} U_\alpha$. The family $(U_\alpha)_{\alpha \in I}$ has *power at most* γ in case $|I| \leqslant \gamma$. If $\mathscr{U}$ is a finite cover we say that $\mathscr{U}$ has *power less than* $\aleph_0$. The family is said to be *open* (respectively, *closed*) in case U_α is open (respectively, closed) for every $\alpha \in I$. When precision is needed, it shall be said that $(U_\alpha)_{\alpha \in I}$ is *open in* X (respectively, *closed in* X).

Now if S is a subset of X and if $(V_\beta)_{\beta \in J}$ is a family of subsets of S, then, by the above, $(V_\beta)_{\beta \in J}$ is an open (respectively, closed) cover of S if $J \neq \varnothing$, if $S = \bigcup_{\beta \in J} V_\beta$, and if, for each $\beta \in J$, V_β is an open (respectively, closed) subset of S in its relative topology. Thus if $(V_\beta)_{\beta \in J}$ is an open cover of S, then $V_\beta \subset S$ for each $\beta \in J$.

As was mentioned above, the closure preserving property of a family of subsets of a set is very useful. For finite families of subsets this property always holds. Along with several other useful facts the following proposition establishes this fact for locally finite families of sets.

1.4 Proposition *Suppose that* $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ *is a family of subsets of a topological space* X.

(1) *The family* $(U_\alpha)_{\alpha \in I}$ *is locally finite if and only if* $(\operatorname{cl} U_\alpha)_{\alpha \in I}$ *is locally finite.*

(2) *If* $\mathscr{U}$ *is locally finite then* $\mathscr{U}$ *is closure preserving.*

(3) *The family* $\mathscr{U}$ *is discrete if and only if* $\mathscr{U}$ *is closure preserving and the elements of* $\mathscr{U}$ *have pairwise disjoint closures.*

(4) *If* $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ *is locally finite, then the collection* $\mathscr{U}^F$ *is locally finite.*

Proof. The facts here are basic and their proofs are straightforward. Consequently many of them are left as exercises. Here statement (2) is shown as a pattern for the proofs. It is clear that

$$\bigcup_{\alpha \in I} \operatorname{cl} U_\alpha \subset \operatorname{cl}\Big(\bigcup_{\alpha \in I} U_\alpha\Big).$$

Conversely suppose that $x \notin \bigcup_{\alpha \in I} \operatorname{cl} U_\alpha$. Since $\mathscr{U}$ is locally finite, there exist a neighborhood W of x and a finite subset K of I such that $W \cap U_\alpha = \varnothing$ if $\alpha \notin K$. Since $x \notin \operatorname{cl} U_\alpha$ for all $\alpha \in I$, certainly, for each $\alpha \in K$, there exists a neighborhood G_α of x such that $G_\alpha \cap U_\alpha = \varnothing$. Let $G = W \cap (\bigcap_{\alpha \in K} G_\alpha)$ and note that G is a neighborhood of x whose intersection with $\bigcup_{\alpha \in I} U_\alpha$ is empty. Hence $x \notin \operatorname{cl}(\bigcup_{\alpha \in I} U_\alpha)$. □

In considering the collection of covers on a given space, it is seen that there is a natural preorder to be defined on it. But first let us recall the definition of a preorder.

1.5 DEFINITION A binary relation $*$ on a set A is a *preorder* if it is *reflexive* ($a * a$ for all a in A) and *transitive* ($a * b$ and $b * c$ implies $a * c$ for all a, b and c in A). A set with a preorder is called a *preordered set*. Usually a preorder $*$ is denoted by $\leqslant$. A preorder on A is called a *partial order* if for a and b in A, $a \leqslant b$ and $b \geqslant a$ implies $a = b$. We say $a < b$ if $a \leqslant b$ and $a \neq b$.

A subset B of a preordered set A is a *chain* if for every pair a and b of elements in B either $a \leqslant b$ or $b < a$. A partially ordered set that is also a chain is called a *totally ordered set*. A partially ordered set W is called *well ordered* if every non-empty subset of W has a first element (that is, for each $B \subset W$ there exists $\beta \in B$ such that $\beta \leqslant b$ for all $b \in B$). A subset B of A is *bounded below* (respectively, *above*) if there is an $\alpha \in A$ such that for each $b \in B$ either $\alpha < b$ (respectively, $b < \alpha$) or $\alpha = b$. The element α is called a *lower* (respectively, *upper*) *bound* for B. An element $\alpha \in A$ is *maximal* (respectively, *minimal*) *in* A if there is no member a in A such that $\alpha < a$ (respectively, $a < \alpha$).

One of the fundamental theorems of set theory is the following.

1.6 THEOREM *The following statements are equivalent.*

(1) (*Axiom of choice*) *Given any non-empty family* $(X_\alpha)_{\alpha \in I}$ *of non-empty pairwise disjoint sets there exists a set* S *consisting of exactly one element from each* X_α.

(2) (*Zermelo's theorem*) *Every set can be well ordered.*

(3) (*Zorn's lemma*) *If each chain in a partially ordered set has an upper bound, then there is a maximal element of the set.*

1.7 DEFINITION Suppose that X is a non-empty set. If $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ and $\mathscr{V} = (V_\beta)_{\beta \in J}$ are two families of subsets of X, then $\mathscr{U}$ is a *refinement of* $\mathscr{V}$ (or $\mathscr{U}$ *refines* $\mathscr{V}$), written $\mathscr{U} < \mathscr{V}$, if $I \neq \varnothing$, if $\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\beta \in J} V_\beta$ and if each element of $\mathscr{U}$ is a subset of some element of $\mathscr{V}$. It is said that $\mathscr{U}$ is a *screen of* $\mathscr{V}$ (or that $\mathscr{U}$ *screens* $\mathscr{V}$ in case $I = J$ and $U_\alpha \subset V_\alpha$ for all $\alpha \in I$). If X is now a

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topological space then the family $\mathscr{U}$ is a *strong screen* of $\mathscr{V}$ if $J = I$ and if $\operatorname{cl} U_\alpha \subset V_\alpha$ for all $\alpha \in I$. If $\mathscr{U}$ is a screen of $\mathscr{V}$ then we say that $\mathscr{V}$ is an *expansion* of $\mathscr{U}$. The family $(U_\alpha)_{\alpha\in I}$ is said to have the *finite intersection property* (respectively, *countable intersection property*) if for every finite (respectively, countable) subset F of I, $\bigcap_{\alpha\in F} U_\alpha \neq \varnothing$. If $n \in \mathbf{N}$, we say that $\mathscr{U}$ has *finite order n* if whenever J is a subset of I such that $\bigcap_{\alpha\in J} U_\alpha \neq \varnothing$ then $|J| \leqslant n$.

Thus a refinement of a cover is itself a cover. Notice also that a refinement of a cover may contain more sets than the given cover. The following lemma demonstrates that when this happens the indexing sets may be made more precise.

1.8 Lemma *Suppose that S is a subset of a topological space X and that $\mathscr{V} = (V_\beta)_{\beta\in J}$ is a locally finite open cover of X whose trace on S refines the open cover $\mathscr{U} = (U_\alpha)_{\alpha\in I}$ of S. Then there exists a locally finite open cover $\mathscr{W}$ of X such that $\mathscr{W}|S$ is a screen of $\mathscr{U}$.*

Proof. Let $\pi: J \to I$ be a function such that $V_\beta \cap S \subset U_{\pi(\beta)}$ for each $\beta \in J$. For each $\alpha \in I$, let

$$W_\alpha = \bigcup_{\beta\in\pi^{-1}(\alpha)} V_\beta.$$

Observe that $\mathscr{W} = (W_\alpha)_{\alpha\in I}$ is an open cover of X such that $W_\alpha \cap S \subset U_\alpha$ for each $\alpha \in I$. To see that $\mathscr{W}$ is locally finite, let x be any element of X. Since $\mathscr{V}$ is locally finite, there exists a neighborhood G of x and a finite subset K of I such that $G \cap V_\beta = \varnothing$ if $\beta \notin K$. Since π is a function, the set $\pi(K)$ is finite. Furthermore, if $\alpha \notin \pi(K)$, then $G \cap W_\alpha = \varnothing$ and it follows that $\mathscr{W}$ is locally finite. □

As hinted above, the important aspect of this lemma is that it will assist us in eliminating indexing sets that are not the 'right size'. In other words, if one has an open cover $\mathscr{U}$ of a subspace S of X and an open cover $\mathscr{V}$ of X such that $\mathscr{V}|S$ refines $\mathscr{U}$, then it is always possible to find another cover of X with the same indexing set as $\mathscr{U}$ which will do the same thing.

So far collections of subsets of a non-empty topological space X have been under consideration. Other collections which should be considered are collections of real-valued functions on X. These are playing an ever more important role in modern topology. For example they can be utilized to give characterizations of some of the topological separation axioms. More will be said about this in Section 2.

Let us record now for the sake of completeness and for reference the topological separation axioms. The T_1-separation axiom will not be a part of the definition of a completely regular space, normal space, etc. as required by some authors (for example, see [XI]).

1.9 Definition A topological space X is a *T_1-space* if for each $x \in X$, $\{x\}$ is closed. A space X is a *Hausdorff space* if for each pair of distinct points x and y in X there are disjoint open sets U and V such that $x \in U$ and $y \in V$. It is a *regular space* if for each $x \in X$ and each closed set F with $x \notin F$, there are disjoint open sets U and V such that $x \in U$ and $F \subset V$. It is a *completely regular space* if for each $x \in X$ and each closed set F with $x \notin F$ there is a real-valued continuous function f on X such that $f(x) = 0$ and $f(y) = 1$ for every $y \in F$. We say that X is a *Tychonoff space* if X is a completely regular T_1-space. The space X is said to be a *normal space* if for each pair F_1 and F_2 of disjoint closed sets there are disjoint open sets U_1 and U_2 such that $F_i \subset U_i$ $(i = 1, 2)$. We say that X is *hereditarily normal* if every subset of X is normal. The space is *perfectly normal* if X is normal and if every closed subset of X is a G_δ.

Now suppose that X is a topological space and that γ is an infinite cardinal number. The space X is said to be *γ-collectionwise normal* if for every discrete family $(F_\alpha)_{\alpha \in I}$ of closed subsets of X of power at most γ there is a family $(G_\alpha)_{\alpha \in I}$ of pairwise disjoint open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in I$. The space X is said to be *collectionwise normal* if for every discrete family $\mathscr{F} = (F_\alpha)_{\alpha \in I}$ of closed subsets of X there is a family $\mathscr{G} = (G_\alpha)_{\alpha \in I}$ of pairwise disjoint open subsets of X such that $F_\alpha \subset G_\alpha$ for every $\alpha \in I$ (that is, $\mathscr{F}$ is screened by $\mathscr{G}$). We say that X is *hereditarily collectionwise normal* if every subset of X is collectionwise normal.

The various notions of 'compactness' described above may now be defined and some comment on the interrelationships between these concepts clarifies their status as useful generalizations.

1.10 DEFINITION Suppose that X is a topological space and that γ is an infinite cardinal number. It is said that X is a *compact space* if every open cover of X has a finite *subcover* (that is, if $(U_\alpha)_{\alpha \in I}$ is an open cover of X then there exists a finite set $F \subset I$ such that $(U_\alpha)_{\alpha \in F}$ covers X). By a *compactification*† of X we mean a compact Hausdorff space in which X is dense (up to homeomorphism). The space X is *countably compact* if every countable open cover of X has a finite subcover. We say that X is *locally compact* if every point of X has a compact neighborhood. The space is *zero-dimensional* if there is a base for the topology consisting of closed and open subsets of X. It is said to be a *Lindelöf* space if every open cover of X has a countable subcover. The space X is *γ-paracompact* if every open cover of X of power at most γ has a locally finite open refinement. The space X is *paracompact* if every open cover of X has a locally finite open refinement. If X is $\aleph_0$-paracompact then it is said that X is *countably paracompact*.

This definition of paracompactness is the one given by Kuratowski [xx]. It differs from the original definition given by Dieudonné [95] in that Dieudonné requires a paracompact space to be a Hausdorff space. Here Kuratowski's definition has been adopted because it is desired that a pseudometric space be paracompact. A proof of this was given by Stone in [359] and it can be found in Chapter 5 of Kelley's book (see [xviii]). There it is also shown that a paracompact regular space is normal and that a paracompact Hausdorff space is regular. Various characterizations of paracompactness have been given. Kelley mentions those due to E. Michael, A. H. Stone, J. S. Griffin and himself. These various characterizations will be used from time to time.

Finally let us mention here that it is this concept of paracompactness, more than any other, that led to the solution of

† We prefer to identify the space X with its homeomorphic copy as opposed to constantly referring to a compactification as an ordered pair (see [xviii]).

the metrizability problem, that is, to the necessary and sufficient conditions for a topological space to be metrizable (see [300], [347], [50]). In Chapter VI, we will see how this concept also leads to other interesting spaces.

Since the notions related to paracompactness and paracompactness itself are important for this treatise, it is important to mention the answers for paracompactness, to some of the questions usually asked about a topological property. Since pseudometric spaces are paracompact, metric spaces are paracompact Hausdorff spaces. Regular Lindelöf spaces are also paracompact. The real line with the topology defined by taking as a subbase for the open sets, the half open intervals, is a regular Lindelöf space. The product of this space with itself is not normal and hence not paracompact. Hence paracompactness is not preserved by taking topological products. Likewise subspaces and quotient spaces of paracompact spaces are not in general paracompact. In the case of subspaces, however, paracompactness is hereditary to closed subspaces. Another example of a non-paracompact space is the space of all ordinals less than the first uncountable ordinal with the order topology. (This example will be discussed in greater detail in Section 9.)

Probably the only concept in 1.10 that is not familiar to the reader is that of γ-collectionwise normality and collectionwise normality. The last concept was introduced by Bing in [50] when he gave his solution to the metrization problem. Later it will be shown that every paracompact space is collectionwise normal. Collectionwise normal spaces have become more interesting in recent years because of their relation to the extension theory of functions and pseudometrics. First then let us give a useful characterization of normality and then relate γ-collectionwise normality for $\gamma = \aleph_0$ to normality.

1.11 PROPOSITION *If X is a topological space, then the following statements are equivalent.*

(1) *The space X is normal.*

(2) *If F is a closed subset of X and if U is an open subset such that $F \subset U$, then there exists an open subset V of X such that $F \subset V \subset \mathrm{cl}\, V \subset U$.*

The easy proof is elementary but should be established by the reader.

The equivalence of $\aleph_0$-collectionwise normality to normality is given in the next theorem.

1.12 THEOREM *A topological space X is a normal space if and only if X is $\aleph_0$-collectionwise normal.*

Proof. Suppose that X is a normal space and let $(F_n)_{n \in \mathbf{N}}$ be a discrete countable family of closed subsets of X. Then $A_1 = \bigcup_{n=2}^{\infty} F_n$ is a closed set (see 1.4) and clearly F_1 is disjoint from it. By 1.11 there exists an open set G_1 of X such that

$$F_1 \subset G_1 \quad \text{and} \quad (\operatorname{cl} G_1) \cap A_1 = \varnothing.$$

The family $(\operatorname{cl} G_1, F_2, F_3, \ldots)$ is a countable discrete family of closed subsets of X. Hence $\operatorname{cl} G_1 \cup \left(\bigcup_{n=3}^{\infty} F_n\right) = A_2$ is a closed set and there exists an open set G_2 of X such that $F_2 \subset G_2$ and $\operatorname{cl} G_2 \cap A_2 = \varnothing$. Then $(\operatorname{cl} G_1, \operatorname{cl} G_2, F_3, F_4, \ldots)$ is a countable discrete family of closed sets. Appealing to finite induction a family $(G_n)_{n \in \mathbf{N}}$ of pairwise disjoint open sets is obtained such that $F_n \subset G_n$ for all $n \in \mathbf{N}$. Hence X is $\aleph_0$-collectionwise normal. Since the converse is obvious, the proof is now complete. □

An important concept to be defined later is that of a uniform space, the natural generalization of a metric space where 'uniform continuity' and 'completeness' may be discussed. Tukey in [XXVIII] gives a careful exposition of the concept by means of the concepts next to be defined. It is interesting to note that covers and refinements of covers play an important role in the theory of uniform spaces also. Hence the role of covers in topology appears as a good concept for unifying others.

1.13 DEFINITION Let $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ and $\mathscr{V} = (V_\beta)_{\beta \in J}$ be two covers of a set X. If $A \subset X$, then by the *star of A with respect* to $\mathscr{U}$ is meant $\cup \{U_\alpha : \alpha \in I \text{ and } U_\alpha \cap A \neq \varnothing\}$. The star of A with