Introduction to categories, homological algebra and sheaf cohomology

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Contents

Preface

Categories and functors, it has often been repeated, were introduced thirty years ago by Eilenberg and MacLane [11] to understand and study certain constructions in algebraic topology. It was soon realized that they provided a useful language in which to treat large tracts of mathematics, ranging as far afield as algebraic geometry on the one hand [19] and automata theory [10] on the other. Thus category theory was developed with the specific needs of certain of these fields in mind. Indeed, it is fair to say that many of the most significant contributions came from mathematicians, expert in one or another area, who forged the new theory to their own use. But as the discipline gained momentum, it started generating internal problems of its own, and an ever increasing band of mathematicians who worked on them became known as categorists. In this respect the situation resembles that of group theory. After people had been working with permutation groups, substitution groups, transformation groups for decades, the notion of 'abstract groups' evolved during the third quarter of last century. This general concept rapidly made clear why the older theories had many features in common. In time, however, questions began to be asked in pure group theory which, as everyone knows, were not always easy to answer.

In this book we do not lose sight of the origins of the subject: categories are there to make different topics more transparent by revealing common underlying patterns. This is particularly true of the notion of adjoint functor which is introduced at an early stage and remains a central theme throughout the book. In view of applications, we have also stuck to the traditional description of a category as consisting of objects and morphisms, rather than as just morphisms with certain operations, sometimes favoured by 'pure' categorists.

The material in the first two chapters is mostly standard, but the arrangement perhaps is not. In chapter 1 representable and adjoint functors straightaway take the stage and are used in our treatment of products and limits. The latter owes much to Lambek [23]. Throughout, many examples and exercises should convince the reader that he is doing 'real' mathematics – albeit a rather superficial part of it – and not just building castles in the air. Only in chapter 2 do we get acquainted with monomorphisms and epimorphisms, kernels and cokernels, which feature right at the beginning of most texts, and we gradually introduce more structure in our categories. Thus we pass from additive and exact categories to abelian and Grothendieck categories. We could not resist presenting the pretty juggling of axioms defining an abelian category, mainly following D. Puppe [34]. Our treatment of Grothendieck categories is frankly utilitarian, geared to the needs of homological algebra; our account has benefited from Popescu [33].

Homological algebra also arose out of algebraic topology when its practitioners began to consider homology groups rather than just Betti numbers. Essentially it deals with derived functors; the treatise by Cartan–Eilenberg [9] was followed by those of MacLane [25] and Hilton–Stammbach [20]. Chapter 3 presents the elements of that theory, but without going at all into applications. As opposed to these books, where the theory is set up for modules and it is then remarked as an afterthought that it also carries through for abelian categories, we work with these straightaway as in Mitchell's book [29]. We first present the theory of Yoneda extensions in a given abelian category. From these we build a large new category and by extending a given functor from the original category to the new category we obtain its sequence of satellites in one fell swoop. Thus the Kan extension theorem yields an existence theorem for satellites. This elegant method was suggested by P. Gabriel in his review of Mitchell's book [15]. I am grateful to him for telling me about it back in 1968. The large category involving the Ext's has the additional advantage that the additivity of these functors follows very easily, a fact also noticed by Brinkmann [7] in a similar setting. Our treatment of derived functors is more conventional; it follows the lines laid down by Grothendieck [18]. We do not discuss spectral sequences.

The fourth and final chapter deals with sheaves and their cohomology. This is an important topic in its own right, but also one in which adjoint functors are employed to great advantage. The cohomology of sheaves of modules displays the techniques developed in the chapter on homological algebra. Applications of sheaf cohomology are manifold, in various fields of mathematics. They are not touched upon here; only the elements of the theory are presented. For a more extensive treatment the reader is referred to the monographs [17], [42] and [6].

In some recent books on categories, the author explains in his preface that he intends to write a textbook as well as a work of reference, for students as well as mature mathematicians. This makes four objectives in all, which seems a tall order to fill. So let me state explicitly that this book is meant as a textbook, not a monograph, treatise or work of reference. I have had in mind students rather than mature mathematicians, learners rather than experts. Even so, the wisdom of publishing as a book notes from a course given seven years ago is legitimately open to question. The subject has rapidly developed in the meantime, but I believe that most of the material in this book should still be considered basic. To my mind, the most important developments have been nonabelian homological algebra and the theory of topoi. In both fields, an authoritative treatise still remains to be written; see however [35], [1], and [43] respectively. For both subjects, certain parts of this book form a useful if not absolutely necessary preliminary.

As already mentioned, the book arose out of a course, given at Utrecht University during the first semester of 1968/1969, followed by a seminar. Notes of the course were taken by A. G. van Asch and W. L. J. van der Kallen. In the seminar, S. H. Nienhuys-Cheng and J. W. Nienhuys exposed sheaves and their cohomology. Notes of their lectures were taken by W. H. Hesselink. To all these people the original Dutch notes, put out by Utrecht University in 1970, owe much. Dr Hesselink moreover has helped considerably with the revision of the fourth chapter for the present edition. My colleague Dr C. J. Penning of the University of Amsterdam undertook the translation. However, his contribution has been far greater than the rendering of the text into English. He made many suggestions and revisions and the final form was decided upon during frequent discussions, in which he often managed to boost my flagging morale. Finally I wish to thank J. Lambek for urging me to publish these notes in the first place and for insisting when I remained reluctant; F. Oort for discouraging and P. Gabriel for encouraging the project. All these people share in the merits, if any, of the final result; but only the author is to blame for its shortcomings. And now, gentle reader, bring along an open mind and judge for yourself.

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