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978-0-521-09525-9 - Introduction to Categories, Homological Algebra and Sheaf Cohomology
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General concepts

1.1 Categories

1.1.1 Definition A *category* **C** is a system of morphisms and objects. We say that f is a *morphism in* **C** *from the object A to the object B* and write $f: A \to B$ or $A \xrightarrow{f} B$. The following conditions should be satisfied.

(i) For each morphism f in **C** there are unique objects A and B in **C** such that $f: A \to B$.

(ii) For each pair of objects A and B in **C** the class of morphisms f such that $f: A \to B$ is a set (A, B). This set may be empty.

(iii) For all objects A, B and C in **C** there is a mapping (called *composition* or *product*) $(B, C) \times (A, B) \to (A, C)$ which assigns to a pair $\ulcorner g, f \urcorner$, with $g \in (B, C)$ and $f \in (A, B)$, the product $gf \in (A, C)$.

(iv) Existence of identities: for every object A in **C** there is a morphism $1_A: A \to A$ with the property that for every object C in **C** and for every couple of morphisms $f: A \to C$ and $g: C \to A$ we have $f 1_A = f$ and $1_A g = g$.

(v) Associativity: for objects A, B, C and D and morphisms $f: A \to B$, $g: B \to C$ and $h: C \to D$ in **C** we have $(hg)f = h(gf)$.

Comments This definition is abstracted from the case that objects are sets and morphisms are mappings. In the abstraction objects are not necessarily sets, nor are morphisms necessarily mappings. The morphisms are the essential ingredients of the theory; the objects are of minor importance.

ad (i). Whether the statement $f: A \to B$ is true or untrue is given together with the category. It is a statement within this category.

ad (ii). Set theory is taken to be known, in particular the difference between class and set. We review this point briefly. A class is not a set if it is bigger than every set. For instance the class of all sets is larger than every set and hence is not a set. If a class K is not larger than a given set X, then K itself is a set. Big sets can be constructed by taking the cartesian product $\prod_{i \in I} A_i$ of sets A_i, where the index set I is also a set. In this book we take a rather naive view of these matters since we are not dealing with foundations. A more careful discussion may be found in [26].

ad (iii). When there is danger of confusion we sometimes specify the category in which morphisms are considered and write $(A, B)_{\mathbf{C}}$. For similar reasons we occasionally write the composition of f and g as $g \circ f$.

ad (iv). The identity morphisms are unique. For the existence of two such identity morphisms 1_A and e_A for an object A implies $1_A = 1_A e_A = e_A$.

Notation Instead of 'A is an object in the category **C**' we write $A \in \mathbf{C}$. This does not therefore have the meaning it has in set theory, since **C** need not be a set.

1.1.2 Examples of categories (*a*) **Sets.** This is the system consisting of sets and mappings. We agree that for each set X there is a unique map going from the void set $\varnothing$ to X. Caution: for $Y \subset Z$ we distinguish between $f: X \to Y$ and $g: X \to Z$, even when $f(x) = g(x)$ for all elements x of X, in order to comply with axiom (i).

(*b*) **Sets$_*$.** The category of sets with base-points. Objects are nonvoid sets V with a given point $*_V$. Morphisms are mappings that map base-points to base-points.

(*c*) **Top.** This is the category of topological spaces. Objects are topological spaces and morphisms are continuous mappings. **Hausd** is the category of Hausdorff spaces.

(*d*) **Top$_*$.** As above, with base-points.

(*e*) **Gr.** Groups with group-homomorphisms.

(*f*) **Ab.** Abelian groups with group-homomorphisms.

(*g*) $\mathbf{V}_k$. The category of vector spaces over a given field k with linear mappings.

(*h*) **Rg**. This is the category of rings with ring-homomorphisms. Rings are supposed to have an identity element and ring-homomorphisms are supposed to map the identity element to the identity element. The smallest ring consists of only one element.

(*i*) **CRg.** Commutative rings and ring-homomorphisms.

(*j*) $\mathbf{M}_R$. R is a fixed ring. This is the category of right R-modules. The morphisms are R-linear mappings. The modules should be right unitary: $x1 = x$ for all elements x of $M \in \mathbf{M}_R$. Analogously, ${}_R\mathbf{M}$ is the category of left R-modules.

(*k*) **CR-alg.** This is the category of commutative R-algebras with algebra-homomorphisms. The algebras are supposed to have an identity

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element and the algebra-homomorphisms should transform identity element to identity element.

(l) **TopGr.** Topological groups and continuous group-homomorphisms.

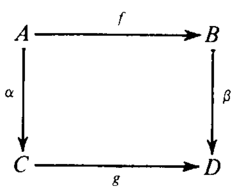
(m) Let I be a preordered class. A category $\mathbf{I}$ is constructed by taking the elements of I as objects. The set of morphisms (i, j) from i to j is empty unless $i \leqslant j$ in which case (i, j) is the set consisting of one element.

(n) Let G be a group. The category $\mathbf{G}$ is the category with single object G and with morphisms all left multiplications.

(o) Let $\mathbf{C}$ be a category. The *dual category* $\mathbf{C}^\circ$ is defined as follows: the objects and morphisms of $\mathbf{C}$ and $\mathbf{C}^\circ$ are the same but the morphisms of $\mathbf{C}^\circ$ run 'in the opposite direction' (arrows are reversed); in other words, for every pair of objects A and B we have $(A, B)_\mathbf{C} = (B, A)_{\mathbf{C}^\circ}$. For $f \in (A, B)_\mathbf{C}$ we write $f^\circ \in (B, A)_{\mathbf{C}^\circ}$. In the case when $g \in (B, C)_\mathbf{C}$, composition in $\mathbf{C}^\circ$ is defined by $f^\circ g^\circ = (gf)^\circ$.

(p) Let $\mathbf{A}$ and $\mathbf{B}$ be categories. The *product category* $\mathbf{A} \times \mathbf{B}$ is defined as follows. Objects are pairs $\ulcorner A, B \urcorner$ of objects with $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Morphisms are pairs $\ulcorner f, g \urcorner$ of morphisms with f a morphism in $\mathbf{A}$ and g in $\mathbf{B}$. The product of any number of categories is defined similarly.

(q) Let $\mathbf{C}$ be a category. One defines a category $\mathbf{C}^2$ in the following way. Objects of $\mathbf{C}^2$ are the morphisms of $\mathbf{C}$. Morphisms of $\mathbf{C}^2$ are certain pairs of morphisms of $\mathbf{C}$. For $f: A \to B$ and $g: C \to D$ in $\mathbf{C}$, the pair $\ulcorner \alpha, \beta \urcorner$ is a morphism from f to g in $\mathbf{C}^2$ if and only if $\alpha: A \to C$ and $\beta: B \to D$ make the following diagram commutative (i.e. $\beta f = g\alpha$):



Note that $\mathbf{C}^2$ is not the same category as $\mathbf{C} \times \mathbf{C}$.

1.1.3 Terminology For $f: A \to B$ in a category $\mathbf{C}$, A is called the *domain* of f and B the *range* of f. We call f an *isomorphism* (notation $f: A \xrightarrow{\sim} B$) and A and B are called *isomorphic* (notation $A \simeq B$) provided there is a morphism $g: B \to A$ in $\mathbf{C}$ such that $fg = 1_B$ and $gf = 1_A$. Given f, such a morphism g is necessarily unique.

A category is called *small* provided the class of objects is a set. In this case the class of all morphisms is also a set since this class equals $\bigcup_{A,B \in \mathbf{C}} (A, B)$ which is a set.

A category is called *concrete* provided its objects are sets endowed with a certain structure which is conserved by morphisms. Examples (a) to (l) are concrete categories. A precise definition will be given in the next section.

$\mathbf{C}'$ is called a *subcategory* of a category $\mathbf{C}$ provided:

(i) $C' \in \mathbf{C}' \Rightarrow C' \in \mathbf{C}$ for all C';

(ii) $(A, B)_{\mathbf{C}'} \subset (A, B)_{\mathbf{C}}$ for all $A, B \in \mathbf{C}'$;

(iii) $(1_{C'})_{\mathbf{C}'} = (1_{C'})_{\mathbf{C}}$.

$\mathbf{C}'$ is called a *full subcategory* of $\mathbf{C}$ provided it is a subcategory with the stronger condition:

(ii)′ $(A, B)_{\mathbf{C}'} = (A, B)_{\mathbf{C}}$ for all A and $B \in \mathbf{C}'$.

For example **Ab** is a full subcategory of **Gr**. The category of all metric spaces with isometries is not a full subcategory of **Top** but **Hausd** is.

1.2 Functors

1.2.1 Definition A *covariant functor* T from a category $\mathbf{C}$ to a category $\mathbf{D}$ is a prescription which assigns to each object $C \in \mathbf{C}$ an object $TC \in \mathbf{D}$ and to each morphism $f \in (A, B)_{\mathbf{C}}$ a morphism $Tf \in (TA, TB)_{\mathbf{D}}$ such that the following conditions are satisfied:

(i) for all $C \in \mathbf{C}$, $T1_C = 1_{TC}$;

(ii) $T(gf) = TgTf$ for all $f \in (A, B)_{\mathbf{C}}$ and all $g \in (B, C)_{\mathbf{C}}$.

Notation $T: \mathbf{C} \to \mathbf{D}$ or $\mathbf{C} \xrightarrow{T} \mathbf{D}$.

1.2.2 Examples (a) T: **Top** $\to$ **Sets**. T is the functor that forgets the topological structure (the *forgetful functor*). Continuous maps between topological spaces are now considered just as maps between the underlying sets.

(b) T: **Gr** $\to$ **Sets**$_*$. Similar to (a). In the underlying set TG for $G \in$ **Gr**, take the identity element as the base-point $*$.

(c) Let G and H be groups and let $f \in (G, H)_{\mathbf{Gr}}$. Consider the categories **G** and **H** as described in 1.1.2(n). Then one may define T: **G** $\to$ **H** by $TG = H$ and $T(\lambda_a) = \lambda_{f(a)}$ (λ_a: left multiplication by a in G).

(d) T: **Gr** $\to$ **Gr**. For $G \in$ **Gr** let $TG = [G, G]$ (commutator subgroup of G) and for $f \in (G, H)_{\mathbf{Gr}}$ let Tf be the restriction of f to $[G, G]$.

(e) T: **Gr** $\to$ **Ab**. For $G \in$ **Gr** let $TG = G/[G, G]$ and for $f \in (G, H)_{\mathbf{Gr}}$ let Tf be defined by $Tf(a[G, G]) = f(a)[H, H]$, $a \in G$.

(f) T: **Top**$_*$ $\to$ **Gr**. For $(X, *) \in$ **Top**$_*$, $T(X, *) = \pi(X, *)$ (fundamental group of X with respect to the base-point $*$). See [39, 1.8].

(g) T: **Top** $\to$ **Ab**. $T = H_n$ (n^{th}-singular homology functor). See [39, 4.4].

(*h*) Let $R \in \mathbf{Rg}$ and $N \in \mathbf{M}_R$. $T: {}_R\mathbf{M} \to \mathbf{Ab}$ is defined by $TM = N \otimes_R M$ and for $f: M \to L$ in ${}_R\mathbf{M}$ by $Tf = 1 \otimes f: N \otimes_R M \to N \otimes_R L$. If R is commutative there is no distinction between left and right modules. In that case $N \otimes_R M$ is an R-module and one may consider T as a functor from ${}_R\mathbf{M}$ to ${}_R\mathbf{M}$.

1.2.3 Terminology A functor $T: \mathbf{C} \to \mathbf{D}$ is called *faithful* provided $Tf = Tg$ implies $f = g$; i.e. for all $A, B \in \mathbf{C}$ the mapping $T: (A, B)_{\mathbf{C}} \to (TA, TB)_{\mathbf{D}}$ is injective. If all these mappings are surjective, the functor is called *full*. T is called an *embedding* provided T is faithful and $TA = TB$ implies $A = B$. A category $\mathbf{C}$ is called *concrete* provided there is a faithful functor $T: \mathbf{C} \to \mathbf{Sets}$. This makes precise the description of a concrete category in 1.1.3.

Let $\mathbf{C}$ be any category and $C \in \mathbf{C}$. Consider the covariant functor $h_C: \mathbf{C} \to \mathbf{Sets}$ defined by $h_C A = (C, A)$ for $A \in \mathbf{C}$ and $h_C f(u) = fu$ for $f \in (A, B)_{\mathbf{C}}$, $u \in (C, A)_{\mathbf{C}}$. This functor is basic in category theory, since it describes the composition of morphisms in the given category.

1.2.4 Definition A *contravariant functor* T from a category $\mathbf{C}$ to a category $\mathbf{D}$ is defined as in 1.2.1 except that now $Tf \in (TB, TA)_{\mathbf{D}}$ and condition (ii) reads $T(gf) = Tf \circ Tg$.

An important example is the contravariant functor (for $C \in \mathbf{C}$) $h^C: \mathbf{C} \to \mathbf{Sets}$ defined by $h^C A = (A, C)$ for $A \in \mathbf{C}$ and $h^C f(u) = uf$ for $f \in (A, B)_{\mathbf{C}}$, $u \in (B, C)_{\mathbf{C}}$.

The contravariant functor $^\circ: \mathbf{C} \to \mathbf{C}^\circ$ introduced in 1.1.2 (*o*) will be denoted by Δ (the dualizing functor). Thus Δ is defined by $\Delta C = C$ and for $f: C \to C'$ by putting $\Delta f = f^\circ: C' \to C$. For any contravariant functor $T: \mathbf{C} \to \mathbf{D}$ one can consider the compositions $T\Delta: \mathbf{C}^\circ \to \mathbf{D}$ and $\Delta T: \mathbf{C} \to \mathbf{D}^\circ$ which are covariant. In this way we often identify a contravariant functor $T: \mathbf{C} \to \mathbf{D}$ with its covariant counterpart $T\Delta: \mathbf{C}^\circ \to \mathbf{D}$. When this does not give rise to confusion we sometimes drop the Δ.

A contravariant functor T is called full, faithful or an embedding functor provided $T\Delta$ is such.

Some more examples of contravariant functors are:

(*a*) $H^n: \mathbf{Top} \to \mathbf{Ab}$ with H^n the n^{th}-cohomology functor. See [28, ch. 5], [39, ch. 6].

(*b*) Spec: $\mathbf{CRg} \to \mathbf{Top}$ (the spectrum of a commutative ring, see [19]).

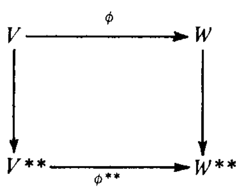
'Functor' on its own usually means a convariant one.

1.2.5 Multifunctors One can also consider functors of more than one variable. Such a functor may be covariant in all variables, contravariant in all variables, or covariant in some and contravariant in the other variables. The reader should write out the formulas for this situation for himself. As a typical example we give the following special case.

Let **C** be any category. For any two objects A and B in **C** denote $(A, B)_{\mathbf{C}}$ by $H\ulcorner A, B\urcorner$. Then H is a *bifunctor* (functor of two variables), contravariant in the first and covariant in the second variable. We consider this as a covariant functor from $\mathbf{C}^{\circ}\times\mathbf{C}$ to **Sets**. Explicitly, if $f^{\circ}: A' \to A$ (i.e. $f: A \to A'$) and $g: B \to B'$, then $H\ulcorner f, g\urcorner: H\ulcorner A, B\urcorner \to H\ulcorner A', B'\urcorner$ is given by $H\ulcorner f, g\urcorner(u) = g \circ u \circ f^{\circ}$ for $u \in H\ulcorner A, B\urcorner$. $H\ulcorner f, g\urcorner$ is also denoted by $(f, g)_{\mathbf{C}}$. We often denote H by $(\text{-}, \text{-})_{\mathbf{C}}$.

1.3 Morphisms of functors

1.3.1 Example Let V be a vector space over a field k and V^{**} its double dual which in our language would be denoted by $((V, k)_{\mathbf{V}_k}, k)_{\mathbf{V}_k}$. There is a particular linear mapping from V to V^{**} that has some remarkable properties which we shall describe now. First let the linear mapping $\hat{}: V \to V^{**}$ be defined by $\hat{v}(f) = f(v)$ for $v \in V$ and $f \in V^*$ (the dual of V). Now let $\phi: V \to W$ be a linear map. Let $\phi^{**}: V^{**} \to W^{**}$ be the corresponding map between the double duals. The following diagram is then commutative:



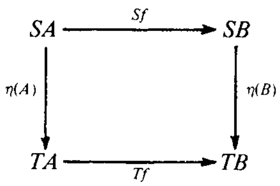
i.e. $\hat{} \circ \phi = \phi^{**} \circ \hat{}$, as the reader may easily check.

1.3.2 Definition Let **C** and **D** be categories and S and T functors from **C** to **D**:

$$\mathbf{C} \underset{T}{\overset{S}{\rightrightarrows}} \mathbf{D}$$

A *morphism* η *from* S *to* T is a class of morphisms $\eta(C)$ in **D**, indexed by the objects C of **C**, such that

(i) $\eta(C): SC \to TC$ in **D** for all $C \in \mathbf{C}$;
(ii) for every $f: A \to B$ in **C** the following diagram is commutative:



Example 1.3.1 shows a morphism ˆ from the identity functor $I: \mathbf{V}_k \to \mathbf{V}_k$ to the 'double dual' functor $**: \mathbf{V}_k \to \mathbf{V}_k$. If one tries to define such a morphism of functors between the identity functor and the 'single dual' functor $*$ one runs into the difficulty that there is no candidate to replace ˆ for this situation. Even if one restricts to the category of finite dimensional vector spaces over k, so that one knows that V and V^* are isomorphic, these isomorphisms depend on the choice of bases and we cannot choose linear maps $\eta(V): V \to V^*$ such that condition (ii) of definition 1.3.2 is satisfied. It is this difference between the mappings $\hat{}(V): V \to V^{**}$ and $\eta(V): V \to V^*$ that motivates the name of 'natural transformation' for the first. It is in this sense that the mysterious word 'canonical' is mostly used. Thus a morphism η of functors is also called a *natural transformation of functors*. As such, this important notion was introduced in the paper of Eilenberg and MacLane [11], which is at the origin of category theory. Indeed, the example of vector spaces and their double duals is theirs.

If for all the objects C of the category $\mathbf{C}$ the morphisms $\eta(C)$ are isomorphisms, the morphism η of functors is called a *natural equivalence* (see 1.3.4).

1.3.3 Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Consider the 'category' $(\mathbf{C}, \mathbf{D})$ whose objects are the covariant functors from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are the functor morphisms defined in 1.3.2. Is $(\mathbf{C}, \mathbf{D})$ a category? It is easily verified that with the obvious composition of morphisms of functors the axioms for a category are satisfied except perhaps axiom (ii). If the category $\mathbf{C}$ is small this axiom is satisfied since for functor morphisms $S, T: \mathbf{C} \to \mathbf{D}$ we have

$$(S, T) \in \prod_{C \in \mathbf{C}} (S(C), T(C))$$

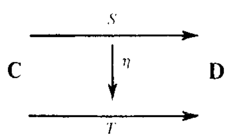
and since the right hand side is a set; see 1.1.1 comment (ii).

If the category $\mathbf{C}$ is not small one sometimes may get around this difficulty as will be seen later (1.9.8 and following).

Although not always justified we still use notation and terminology for $(\mathbf{C}, \mathbf{D})$ as if it were a category, except, of course, in those cases where the missing axiom is essential. Thus we will not use, for instance, the notation h^T for $T \in (\mathbf{C}, \mathbf{D})$ since this would mean $h^T: (\mathbf{C}, \mathbf{D}) \to \mathbf{Sets}$.

Categories allow one to define functors between them. Natural transformations between functors then occur as the morphisms in a 'category' where objects are the functors between two given categories. This 'closure of category theory within itself' is of fundamental importance, both from a foundational point of view, and in the more sophisticated branches of the theory. Brashly, one sometimes speaks of the 'category' **Cat**: its objects are all categories (or possibly all small categories) while $(\mathbf{C}, \mathbf{D})_{\mathbf{Cat}}$ consists of all functors from **C** to **D**.

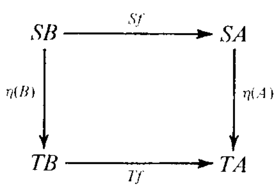
1.3.4 Remark If a morphism η of functors



is such that for every $C \in \mathbf{C}$ the morphism $\eta(C)$ is an isomorphism in the category **D**, the reader may easily establish the fact that then the morphism η has the usual properties required for an isomorphism in the 'category' $(\mathbf{C}, \mathbf{D})$, and vice versa. Therefore a natural equivalence $\eta: S \to T$ is also called an isomorphism from S to T. The functors S and T are called *naturally equivalent* or *isomorphic*. We denote this by $S \simeq T$.

1.3.5 Definition Let S and T be contravariant functors from **C** to **D**. A morphism η from S to T assigns to each $C \in \mathbf{C}$ a morphism $\eta(C)$ such that

(i) $\eta(C): SC \to TC$ for all $C \in \mathbf{C}$;
(ii) for $f: A \to B$ in **C** the following diagram is commutative:

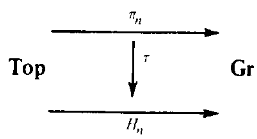


In other words, η is a morphism between the covariant functors $S\Delta$ and $T\Delta$ from $\mathbf{C}^\circ$ to **D**.

1.3.6 Some more examples (*a*) There is a completely analogous morphism of functors $\hat{}: I \to {**}$ from the identity functor to the double dual functor for the case $\mathbf{M}_R$ instead of $\mathbf{V}_k$. As the ring R need not be commutative one has to distinguish between $\mathbf{M}_R$ and ${}_R\mathbf{M}$. For any object $M \in \mathbf{M}_R$ the single dual M^* can be made in the evident and usual way into an object of ${}_R\mathbf{M}$. Thus the double dual M^{**} is again an object of $\mathbf{M}_R$.

(*b*) Let $F: {}_R\mathbf{M} \to {}_R\mathbf{M}$ be the functor that assigns to each left R-module M the free left R-module FM with a base consisting of all nonnull elements of M. This yields a morphism $\varepsilon: F \to I$ from the functor F to the identity functor by mapping each basis element to its underlying element in M.

(*c*) The so-called Hurewicz homomorphism $\tau: \pi_n \to H_n$ (see [28, p. 322], [39, p. 390]) is a morphism of functors:



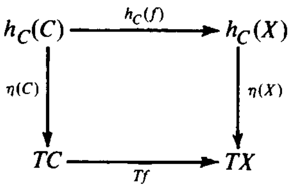
(*d*) Let R be a commutative ring and let T be the bifunctor $\mathring{\mathbf{M}}_R \times \mathbf{M}_R \to \mathbf{M}_R$: $T\ulcorner M, N\urcorner = (M, R) \otimes_R N$, and $T\ulcorner f, g\urcorner(u \otimes y) = uf^\circ \otimes g(y)$. Let H be the bifunctor $\mathring{\mathbf{M}}_R \times \mathbf{M}_R \to \mathbf{M}_R$ defined by $H\ulcorner M, N\urcorner = (M, N)$ and $H\ulcorner f, g\urcorner h = ghf^\circ$ as in 1.2.5. Define $\theta: T \to H$ by $\theta\ulcorner M, N\urcorner(u \otimes y)(m) = u(m)y$ for $u \in (M, R)$, $y \in N$ and $m \in M$. It is left to the reader to verify that θ is well defined and is indeed a morphism from T to H. This is an example of a morphism between bifunctors. More generally one defines morphisms between multifunctors. Keeping certain objects constant one obtains by restriction of the variables new (multi-) functors.

(*e*) Let $T: \mathbf{A} \to \mathbf{B}$ be a functor. Then 1_T is a frequently occurring functor isomorphism from T to T given by $1_T(A) = 1_{TA}: TA \to TA$.

1.4 Representable functors

1.4.1 Example Let $S: \mathbf{Gr} \to \mathbf{Sets}$ be the forgetful functor which assigns to each group G its underlying set SG. Let $\eta: h_Z \to S$ be the morphism defined by $\eta(G)(f) = f(1)$ for $f \in (\mathbb{Z}, G)_{\mathbf{Gr}}$, where 1 is the generator of the infinite cyclic group $\mathbb{Z}$. (For typographical reasons we write h_Z for the functor $(\mathbb{Z}, -)_{\mathbf{Gr}}$.) Define $\lambda: S \to h_Z$ by putting, for $x \in SG$, $\lambda(G)(x) = f$ if f is the group-homomorphism $\mathbb{Z} \to G$ defined by $f(1) = x$. The reader may easily verify that η and λ are morphisms of functors and that $\eta\lambda = 1_S$ while $\lambda\eta = 1_{h_Z}$. In other words, the functors S and h_Z are isomorphic or naturally equivalent as discussed in 1.3.4.

1.4.2 The Yoneda lemma This situation may be generalized. However we first establish the following fact. Let **C** be any category and S and T functors from **C** to **Sets**. As we have already pointed out, (S, T) need not be a set. But, for $C \in \mathbf{C}$, (h_C, T) is a set, which can be seen as follows. Since $\eta: h_C \to T$ is a morphism of functors we have for each $f: C \to X$ the commutative square



Analysing these maps for $1_C \in h_C C$ we find $Tf(\eta(C)(1_C)) = (Tf \circ \eta(C))(1_C) = (\eta(X) \circ h_C f(1_C) = \eta(X)(f)$. Consequently for each X the mapping $\eta(X)$ is entirely determined by the element $\eta(C)(1_C) \in TC$. The latter being a set, so is (h_C, T). This point having been settled we proceed with the main discussion.

Yoneda's lemma There are mappings

$$\Lambda: (h_C, T) \to TC$$

and

$$\Theta: TC \to (h_C, T)$$

such that

$$\Theta\Lambda = 1_{(h_C, T)}$$

and

$$\Lambda\Theta = 1_{TC}.$$

Proof. For $\eta \in (h_C, T)$ let $\Lambda(\eta) = \eta(C)(1_C)$ and for $c \in TC$ let $\Theta(c)(X)(f) = Tf(c)$, for $X \in \mathbf{C}$ and $f \in (C, X)$. We have to check that $\Theta(c) \in (h_C, T)$. It follows from the definition of Θ that $\Theta(c)(X): (C, X) \to TX$; for if $f \in (C, X)$, then $Tf \in (TC, TX)$ and hence $Tf(c) \in TX$. Now let $g: X \to Y$ in **C**, then the following diagram commutes: