HOMOLOGY THEORY

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AN INTRODUCTION TO ALGEBRAIC TOPOLOGY

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CONTENTS

GENERAL INTRODUCTION

This book has been written with the intention of providing an introduction to algebraic topology as it is practised today. The reader is not supposed at the outset to possess any knowledge of algebraic topology; indeed, even the reader with no knowledge of analytic topology or abstract algebra is provided, in the Background to Part I, with a synopsis of the facts that are taken for granted in the text. The treatment throughout has been subject to the consideration that, if the book is to serve its purpose, it must provide an account of the basic notions of algebraic topology intelligible to the mathematician inexperienced in the techniques and problems described. However, though the treatment is elementary, we have been more ambitious in our choice of material than is customary in elementary textbooks. It appears to us that the literature is rich in advanced textbooks and adequate in elementary introductory textbooks, but that the two types of book are not very effectively linked. Again, the advanced textbooks themselves fall into two classes which may broadly be described as classical and modern and the rapid shifts of emphasis which the subject has experienced make it difficult always to recognize classical arguments in their modern dress. We have tried to provide the links which we believe the student might find difficulty in providing for himself from a study of the available literature.

Thus, while our beginning is quite elementary, we have been able, by omitting certain topics, particularly those treated canonically in classical works, to reach in later chapters the parts of the subjects which lie in the immediate foreground of present-day research. The book is divided into two parts. The first part, 'Homology Theory of Polyhedra', also includes a chapter on the fundamental group and covering spaces and a chapter introducing the reader to Homological Algebra. With the exception of certain material in the latter chapter, the notions presented are classical—though the style of presentation is intended to be modern. This part is entirely self-contained with one exception (the proof of the Künneth formula for the homology groups of topological products). Homology groups with arbitrary coefficients and the cohomology‡ ring are defined for pairs of simplicial complexes and proved to be topological invariants—indeed, homotopy

‡ In this book called 'contrahomology' for reasons explained below.

invariants—in the case of finite complexes. The axioms of Eilenberg and Steenrod are verified; and the viewpoint of their textbook as expressed in the (Eilenberg–Maclane) theory of *categories* and *functors* is adopted in the sense that we always bring out the naturality (in its technical sense) of the concepts and transformations introduced. We do not actually define the notions of category and functor in Part I (though the reference to Eilenberg and Steenrod appears several times), but we hope that our emphasis on the ideas appropriate to the functorial approach (e.g. covariance and contravariance, canonical isomorphisms) will prepare the reader for the next stage of his battle with the literature of algebraic topology.

We take this opportunity emphatically to reject the idea that, by attempting to be modern and, in particular, by approximating to the Eilenberg–Steenrod approach, we have made the book harder than it need be. On the contrary, the systematic enunciation of homology properties certainly makes it easier to grasp the pattern of homology theory; this is just a special case of a general principle that the systematic introduction of algebraic procedures into any branch of mathematics advances it. This is certainly not to say that the geometric content of algebraic topology should be suppressed; indeed, we have been at pains to give, where appropriate, geometrical significance to the algebraic concepts.

Part II is concerned with the application of homology theory to the study of general topological spaces. For these purposes, and particularly for chapter 7, the reader needs certain facts from homotopy theory; these are provided in the Background to Part II. Chapter 7 applies the simplicial cohomology theory developed in Part I to the study of maps of polyhedra into arbitrary spaces, and is intended to serve as an introduction to obstruction theory. The next two chapters are devoted to the singular homology theory (there is a brief description of Čech cohomology in an appendix to chapter 8). The singular theory is developed in full detail; the axioms are verified, and canonical isomorphisms established (i) between singular simplicial and singular cubical homology, and (ii) between the singular theory for polyhedra and the simplicial theory of Part I. The gap in Part I is filled by showing that the chain complex used in chapter 5 does, in fact, yield the singular homology groups of the topological product. Chapter 9, on the singular cohomology ring, contains a section on the Hopf invariant (of maps of S^{2n-1} in S^n) and closes with an appendix in which the concept of naturality is, at last, defined precisely. It was plainly necessary to do this at some stage since naturality appears as a working tool in many of our arguments, particularly in chapters 8 and 9. We delayed the definition to this late stage in the belief that its general significance would be more easily understood by a reader already possessed of a number of particular examples.

It is held by some that it is unnecessary to present simplicial homology theory since it coincides with singular theory on the category of polyhedra. It has indeed been trenchantly argued that the only two homology theories of importance are the singular theory (appropriate to homotopy theory) and the Čech theory (appropriate to the study of manifolds and algebraic geometry), and that the simplicial theory is, in a sense, the intersection of these two theories. Our case for devoting a considerable part of the book to the simplicial theory of polyhedra is, in the first place, that the book is explicitly an introduction and simplicial theory is, in our view, the best introduction to homology theory; conceptually it is certainly the easiest to comprehend, in that it is geometrically satisfying and that, for a finite complex, all groups involved are finitely generated. Furthermore, in developing simplicial theory, basic notions common to any homology theory (e.g. chain complex and chain map) are made familiar and techniques appropriate to the more powerful homology theories are exemplified in an elementary setting (e.g. the Eilenberg–Maclane technique of acyclic models generalizes the simpler notion of acyclic carriers). We also argue that it is easier to compute by means of simplicial complexes; indeed, some attention is paid in Part I to modifications of simplicial complexes which make computation easier and quicker (pseudodissections and block chains).

The final chapter is concerned with two abstract developments of homology theory, namely, spectral homology theory and the homology theory of groups. These two topics have come into great prominence in recent years, and a knowledge of the basic facts of these theories seems to be part of the essential equipment of the algebraic topologist. Applications of these theories to topology and algebra are indicated.

Each chapter ends with a batch of exercises. Those for the early chapters are meant mainly to give practice and confidence; those for the later chapters are harder and more interesting.

Certain notational innovations have been made and two of them, at least, need some defence. Throughout the book, the prefix 'co-' which traditionally appears before the terms chain, cycle, boundary homology, to describe the dual notions, is replaced by 'contra-'. We thus adopt the third of three suggestions made by M. M. Postnikov and V. G. Boltyanskii. The preference for 'contra-' over 'co-' is that contra objects transform *contra*variantly. For example, if C is a chain complex and Hom (C, G) the associated contrachain complex (with values in G), then a chain map $\phi : C \to D$ induces a contrachain map $\phi^{\cdot} : \text{Hom}(D, G) \to \text{Hom}(C, G)$; thus $\phi^{\cdot}$ operates in the direction *opposite* to that of ϕ. Thus, from this point of view, the new terminology is more logical; equally logical would be the replacement of the terms chain, cycle, ..., by *co*chain, *co*cycle, ..., since these objects transform *co*variantly (with respect to maps of topological spaces). The latter replacement is the second of the Postnikov–Boltyanskii suggestions, but, to quote a familiar phrase, the time is clearly not ripe for this innovation! Thus the terminology we have adopted is avowedly a half-way house between the familiar terminology and that demanded by logical consistency. Had it been possible to use the terms co-homology and contrahomology as suggested, the term homology would then have been available to describe a theory (of differential graded groups) of which cohomology and contrahomology are the most important manifestations. Despite our unwillingness to adopt the term cohomology as proposed by Postnikov and Boltyanskii, we have used the word homology at times in the sense indicated above,‡ for example, in the title of the book and the titles of chapters 2 and 8.

Our second major innovation is to write transformations of spaces and of 'co-' objects on the *right* and transformations of 'contra-' objects on the *left* (more precisely, maps in the image category of a category of topological spaces under a covariant (contravariant) functor are written on the right (left)). This device would be in-convenient if we had to deal frequently with mixed functors, but this is not the case in the present work where most functors are either covariant or contravariant; where a mixed object is in question (e.g. Hom (A, B) as a functor of A and B), no *a priori* rule is given. On the other hand, the device turns out to have distinct advantages. For spaces, for example, if f maps X to Y and g maps Y to Z, the map f *followed* by the map g appears as fg, not gf. Further, reverting to chain maps ϕ and their associated (adjoint) contrachain maps $\phi^{\cdot}$, it is convenient that $(\phi\psi)^{\cdot} = \phi^{\cdot}\psi^{\cdot}$. Again, for the Kronecker product, we have $(c\phi, d) = (c, \phi^{\cdot}d)$, $c \in C$, $d \in \text{Hom}(D, G)$.

‡ The first of the P–B suggestions, namely, that the word 'homology' should be reserved for free chain complexes would preclude this usage.

We accept the statement that contravariant functors reverse arrows in its literal sense when writing out sequences of objects and maps. Thus, for example, we write that

$$C \xrightarrow{\phi} D \xrightarrow{\psi} E$$

induces $$\operatorname{Hom}(C, G) \xleftarrow{\phi^{\cdot}} \operatorname{Hom}(D, G) \xleftarrow{\psi^{\cdot}} \operatorname{Hom}(E, G).$$

Actually, from chapter 3 onwards, we follow E. C. Zeeman in writing $C \pitchfork G$ for $\operatorname{Hom}(C, G)$. The reason we adopt this notation is that it brings out better the duality between the tensor product, $\otimes G$, and $\pitchfork G$. In order to have the greatest possible uniformity of notation we write $A * B$ for the torsion product of two abelian groups and, again to emphasize duality, $A \dagger B$ for $\operatorname{Ext}(A, B)$.

We have been concerned to try to avoid repeated use of complicated symbols even where the requirements of logical precision appear to demand them. Thus, for example, the p-dimensional chain group, with coefficients in G, of the simplicial pair K, K_0, oriented by the orientation α, may be represented by the cumbersome symbol $C_p^{\alpha}(K, K_0; G)$. Our general principle is to write the complete symbol in the definitions and then, in subsequent appearances of the same concept, to reproduce only as much of the symbol as is not clearly implied by the context. Thus the symbol mentioned above might well be replaced in subsequent occurrences by C_p. Here in the introduction we merely state the principle—in the text of the book we prepare the reader for each specific abbreviation.

The reader will be familiar with the grammatical flexibility of the symbol '$=$'. In the same way the symbol '$f: A \to B$' should normally be read as 'f, *which is* a transformation from A to B', but it will occasionally have to be read otherwise, in particular as 'f *is* a transformation from A to B'. Similar grammatical ambiguity occurs with the symbol '$a \in A$'.

We have adapted Halmos' notation by marking the end of a proof with], having been impressed by its effectiveness in the book *General Topology* by J. L. Kelley. The appearance of] immediately after the statement of a theorem (or proposition, etc.) indicates either that the proof has immediately preceded the statement or that the proof is left to the reader. As a further guide to the reader we have adopted the devices (i) of starring certain sections or parts of sections; and (ii) of putting some material in small type. If a part is starred the reader may conclude that its contents are somewhat more sophisticated than its place in the book would suggest, and that it may be omitted on first reading. The use of small type, on the other hand, indicates only that the material is not central to the logical development of the subject; thus illustrative remarks and examples are often so treated.

Our numbering system is by triples $\alpha.\beta.\gamma$, where α is the chapter‡ number, β the section number within the chapter, and γ the item number within the section. Even in chapter α, $\alpha.\beta.\gamma$ is *not* abbreviated to $\beta.\gamma$, so that a reference to 5.7, for instance, is a reference to a section and not to an item. The items are numbered consecutively throughout a section, whether they be theorems, propositions, lemmas, corollaries, definitions, examples, or just displayed formulae. The main exception to this rule is that, following Eilenberg and Steenrod, we may index the statement in contrahomology corresponding to statement $\alpha.\beta.\gamma$ in homology by the same triple followed by the affix c; another exception occurs in 5.3 as is there explained.

We have had help of various kinds from many people, most of whom can only be thanked in rather general terms. Our mathematical friends have contributed lavishly in conversation and discussion, and we have only made specific acknowledgements in the text for a fraction of the ideas that we have knowingly adopted; there must be many more that we have adopted thinking them to have been our own. Dr E. C. Zeeman read the manuscript of what we had intended to put in the book and by his imaginative and effective criticisms persuaded us to improve it in numberless ways; in particular, many of the robuster examples are due to him. Mr E. C. Thompson has read the book in the less flexible form presented by galley proofs and has been of the greatest help in eliminating mistakes and obscurities. We are very grateful indeed to both of them.

We are grateful also to many people for the stimulus they have provided. Some have done this by the constructive and occasionally embarrassing interest they have shown in the progress of the book. Others, to whom we are quite as grateful, have stimulated us in another way—those of our friends who have told us that this is not the time to write a book on Homology Theory, at any rate not a book of this kind, and those who have taken a less technical line (our wives, in fact) and have made it clear that this is not the time to write a book of any kind whatsoever.

‡ For the Background to Part I, $\alpha = \mathrm{I}$; for the Background to Part II, $\alpha = \mathrm{II}$.

On the production side we have been extremely lucky. Miss Kerridge, who did the bulk of the typing, contrived to make good-looking copy of some very unpromising handwriting and has earned our warmest thanks. So too have the officers and staff of the Cambridge University Press who have been involved in the book; all of them have shown us unfailing helpfulness and patience.

P.J.H
S.W.

February 1960

Note on the Second Impression

A number of misprints, and two mistakes, have been corrected at the second printing. We are grateful to those who have drawn our attention to these; and, in particular, to Professor W. S. Massey for pointing out that the original statement of 9.4.16 was wrong.

P.J.H.
S.W.

March 1962

Note on the Fourth Impression

We are grateful to Dr J. S. P. Wang for suggesting an improvement in Theorem 1.8.5. We have also corrected a few more misprints and again thank those who have drawn our attention to them.

P.J.H.
S.W.

May 1967