PART I

HOMOLOGY THEORY OF POLYHEDRA

BACKGROUND TO PART I

1 Analytic topology

A *topological space* is a set X in which certain subsets, called *open sets*, are distinguished; the collection of open sets satisfies the axioms:

(O 1) the union of any number of open sets is open;
(O 2) the intersection of any finite number of open sets is open;
(O 3) the whole space and the empty set are open.

To prescribe the open sets is to assign a *topology* to the set X. If $\mathscr{U}$, $\mathscr{V}$ are two topologies on the set X, then $\mathscr{U}$ is *finer* than $\mathscr{V}$ ($\mathscr{V}$ is *coarser* than $\mathscr{U}$) if every set of X which is open in the topology $\mathscr{V}$ is open in the topology $\mathscr{U}$. A set of open sets of X forms a *base* (for the open sets) if every open set of X is a union of sets of the base.

A *closed* subset of the topological space X is the complement of an open set; thus a topology is assigned by prescribing the closed sets and the closed sets must satisfy the axioms:

(C 1) the union of any finite number of closed sets is closed;
(C 2) the intersection of any number of closed sets is closed;
(C 3) the whole space and the empty set are closed.

If X_0 is a subset of the topological space X, the *induced topology* in X_0 is that in which the open sets are the intersections with X_0 of the opens sets of X. Subsets will always be supposed to carry the induced topology. Plainly, if $X_1 \subseteq X_0 \subseteq X$, then X_0 and X induce the same topology in X_1, and an open (closed) subset of an open (closed) subset of X is an open (closed) subset of X.

The *interior* of X_0 is the union of all open subsets of X contained in X_0.

I. 1.1 Proposition. *The interior of X_0 is the largest open set contained in X_0.*

The *closure* of X_0 is the intersection of all closed sets containing X_0.

I. 1.2 Proposition. *The closure of X_0 is the smallest closed set containing X_0.*

If $x \in X$, a *neighbourhood* of x in X is a subset of X containing x in its interior.

I. 1.3 Proposition. *The closure of X_0 is the sets of points $x \in X$ such that every neighbourhood of x meets X_0.*

The *frontier* of X_0 in X is the intersection of the closure of X_0 and the closure of its complement. X_0 is *dense* in X if its closure is X. The sequence $x_1, x_2, \ldots, x_n, \ldots$ of points of X *converges* to $x \in X$ if every neighbourhood of x contains all but a finite number of points of the sequence.

The collection $\{X_i\}$ of subsets of X is a *covering* of X if their union is X. The covering is *open* (*closed*) if each X_i is open (closed). The covering is *finite* (*countable*) if there are finitely many (countably many) sets in the covering. The covering $\{Y_j\}$ is a *subcovering* (*refinement*) of the covering $\{X_i\}$ if each Y_j is (is contained in) an X_i.

A *map* $f: X \to Y$ from the space X to the space Y is a *continuous* function‡ from X to Y; the *continuity* of f is expressed by the property that, if U is any open subset of Y then $f^{-1}(U)$, the set of points of X mapped by f into U, is open in X. Equivalently, f is continuous provided $f^{-1}(F)$ is closed whenever F is closed. If $X_0 \subseteq X$, then $X_0 f$ is the set of points xf, $x \in X_0$, and is called the *f-image* of X_0. If $Y_0 \subseteq Y$, $f^{-1}(Y_0)$ is called the *f-counterimage* of Y_0. A map $f: X \to Y$ determines functions $f_0 : X_0 \to Y$, $f' : X \to Xf$ given by $xf_0 = xf$, $x \in X_0$; $xf' = xf$, $x \in X$. We may write f_0 as $f \mid X_0$, and we say that f_0 is the *restriction* of f to X_0 and that f is an *extension* of f_0 to X, or over X.

I. 1.4 Proposition. *The functions f_0, f' are continuous.*

I. 1.5 Proposition. *If $f: X \to Y, g: Y \to Z$ are maps, then $fg: X \to Z$ is a map.*

Let $\{A_i\}$ be a *finite* covering of X by closed sets and let $f_i : A_i \to Y$ be maps such that $f_i \mid A_i \cap A_j = f_j \mid A_i \cap A_j$. Then we may define a unique function $f: X \to Y$ by $f \mid A_i = f_i$.

I. 1.6 Proposition. *The function $f: X \to Y$ is continuous.*

(For if $F \subseteq Y$ is closed then $f^{-1}F = \bigcup_i f_i^{-1}(F)$; but $f_i^{-1}(F)$ is closed in A_i and hence in X so that, by (C 1), $f^{-1}F$ is closed in X and f is continuous.) A similar result holds for arbitrary coverings by open sets.

If $X_0 \subseteq X$, let $i : X_0 \to X$ be given by $xi = x$, $x \in X_0$. Then i is plainly continuous and we call it an *inclusion* map or *injection*. If

‡ All functions are understood to be single-valued.

there exists a map $p : X \to X_0$ such that $ip = 1$, where 1 is the *identity map* of X_0, then X_0 is a *retract* of X and p is a *retraction*.

The map $f : X \to Y$ is a *homeomorphism* of X *onto* Y (abbreviated to *homeomorphism*) if there is a map $g : Y \to X$ (called the *inverse* of f) such that $fg = 1 : X \to X$ and $gf = 1 : Y \to Y$. We write $f : X \cong Y$ and say that X and Y are *homeomorphic* or *of the same topological type*.

I. 1.7 Proposition. *$X \cong Y$ is an equivalence relation.*

The map $f : X \to Y$ is a *homeomorphism* of X *into* Y if the induced map $f' : X \to Xf$ is a homeomorphism. Plainly $i : X_0 \to X$ is a homeomorphism of X_0 into X; if $f : X \to Y$ is a homeomorphism of X into Y, then X may be *embedded* in Y by identifying x with xf, $x \in X$. The map $f : X \to Y$ is *locally homeomorphic* if each $x \in X$ possesses a neighbourhood U such that f maps U homeomorphically onto a neighbourhood of xf.

Let $f : X \to Y$ be a function from the *space* X *onto the set* Y; then the *identification topology on* Y *determined by* f is the topology in which $Y_0 \subseteq Y$ is closed if and only if $f^{-1}(Y_0)$ is closed. If Y is given the identification topology, f is called an *identification map* or *projection*.

I. 1.8 Proposition. *The identification topology is the finest topology consistent with the continuity of f.*

(Note that this proposition asserts, *a fortiori*, that the identification topology *is* a topology.)

Let R be an equivalence relation on the points of X (thus xRx; xRx' implies $x'Rx$; xRx' and $x'Rx''$ together imply xRx''). Let Y be the set of R-equivalence classes and let $k : X \to Y$ send each point to its equivalence class. If Y is given the identification topology determined by k we may write $Y = X/R$ and say that Y is the *quotient space* of X by the relation R (with the *quotient topology*).

Given two spaces X and Y their *topological product* $X \times Y$ is the set of pairs (x, y), $x \in X$, $y \in Y$, with the topology in which a base of open sets consists of the sets $U \times V$, where U is open in X and V is open in Y. The maps $(x, y) \to x$, $(x, y) \to y$ project $X \times Y$ onto X, Y. If $x_* \in X$, $y_* \in Y$, the maps $x \to (x, y_*)$, $y \to (x_*, y)$ embed X, Y in $X \times Y$; we refer to these maps (by abuse of language) as injections.

The subspace $(X \times y_*) \cup (x_* \times Y)$ of $X \times Y$ is called the *bunch* of X and Y and is written $X \vee Y$ if there is no need to specify the points

x_*, y_*. Then $X \times y_*$, $x_* \times Y$ are subspaces‡ of $X \vee Y$ whose union is $X \vee Y$ and whose intersection is the single point (x_*, y_*). Thus, using the injections defined above, we may think of $X \vee Y$ as the 'union of X and Y with a single common point'.

Clearly the notion of topological product extends to finite collections of spaces (it also extends to infinite collections, but we shall only be concerned with finite products).

A *metric* in the set X is a real-valued function ρ defined on (ordered) pairs of points of X and satisfying

(M 1) for $x, y \in X$, $\rho(x,y) \geqslant 0$;
(M 2) $\rho(x,y) = \rho(y,x)$;
(M 3) $\rho(x,y) = 0$ if and only if $x = y$;
(M 4) (*triangle inequality*) for $x, y, z \in X$, $\rho(x,y) + \rho(y,z) \geqslant \rho(x,z)$.

The *ε-ball, centre x*, in the set X (with the metric ρ) is the set, $V(x, \epsilon)$, of points $y \in X$ with $\rho(x,y) < \epsilon$. The *metric topology* in the set X is the topology in which the open sets are unions of sets of the form $V(x, \epsilon)$.

I. 1.9 Proposition. *The metric topology is a topology.*

X is a *metric space* if its topology is given by a metric ρ. The *diameter* of a subset X_0 of a metric space X is $\underset{x, y \in X_0}{\text{l.u.b.}}\ \rho(x,y)$.

The topological space X is *metrizable* if the set X admits a metric such that the metric topology coincides with the given topology.

I. 1.10 Proposition. *A function $f: X \to Y$ from the metric space X to the metric space Y is continuous if and only if, for any $x \in X$ and $\epsilon > 0$, there exists $\delta(x, \epsilon) = \delta > 0$ such that $V(x, \delta) f \subseteq V(xf, \epsilon)$.*

The map f is *uniformly continuous* if δ may be chosen independently of x.

The space X is

Hausdorff if any two distinct points of X possess disjoint neighbourhoods;

compact if every open covering of X has a finite subcovering;

sequentially compact if every sequence of points of X has a convergent subsequence;

locally compact if every point of X has a compact neighbourhood;

connected if X is not the union of two disjoint non-empty closed sets;

separable if X possesses a countable dense subset.

I. 1.11 Proposition. *A metric space is compact if and only if it is sequentially compact.*

‡ They are closed subspaces if the points x_*, y_* are closed sets.

I.1.12 Proposition. *A compact metric space is separable.*

I.1.13 Proposition. *A (continuous) map of a compact metric space into a metric space is uniformly continuous.*

I.1.14 Proposition. *The limit of a convergent sequence in a Hausdorff space is unique.*

I.1.15 Proposition. *A compact subset of a Hausdorff space is closed.*

I.1.16 Proposition. *If X is locally compact and Hausdorff then every neighbourhood contains a compact neighbourhood.*

I.1.17 Proposition. *Let $f: X \to Y$ be a map of the compact space X onto the Hausdorff space Y; then Y has the identification topology determined by f. In particular, f is a homeomorphism if it is* (1, 1).

I.1.18 Proposition. *A subset of Euclidean space R^n is compact if and only if it is closed and bounded.*

Let $r \geqslant 0$ be a real number and let I_r be the closed interval $0 \leqslant t \leqslant r$ in R^1. A *path* in X is a map $f: I_r \to X$, for some $r \geqslant 0$. The path *starts* at $0f$, its *initial* point, and *ends* at rf, its *final* point. X is *path-connected* if, given any two points $x, y \in X$, there exists a path in X starting at x and ending at y. The *component* (*path-component*) of X containing x is the largest connected (path-connected) subset of X containing x.

I.1.19 Proposition. *A path-connected space is connected.*

The *Hilbert space* H^∞ is the metric space whose points are infinite sequences of real numbers $\mathbf{u} = (u_1, u_2, \ldots, u_n, \ldots)$ such that $\sum_{n=1}^{\infty} u_n^2$ converges, the metric being

$$\rho(\mathbf{u}, \mathbf{v}) = \left\{ \sum_{n=1}^{\infty} (u_n - v_n)^2 \right\}^{\frac{1}{2}}.$$

I.1.20 Proposition. *H^∞ is separable.*

(The set of finite sequences of rationals is a countable dense subset.)

Euclidean space R^n (of dimension n) may be embedded in H^∞ by identifying $(u_1, \ldots, u_n)$ with $(u_1, \ldots, u_n, 0, 0, \ldots)$. Then there are inclusions

$$R^1 \subseteq R^2 \subseteq \ldots \subseteq R^n \subseteq R^{n+1} \subseteq \ldots \subseteq H^\infty.$$

The *unit n-sphere* S^n is the subset of R^{n+1} given by $u_1^2 + \ldots + u_{n+1}^2 = 1$. It is the frontier in R^{n+1} of the (closed) ball V^{n+1}, given by

$u_1^2 + \ldots + u_{n+1}^2 \leqslant 1$. An *n-cell* is a homeomorph of V^n; an *n-sphere* is a homeomorph of S^n; the *frontier* of the n-cell E^n is the image of S^{n-1} under a homeomorphism $V^n \to E^n$; the complement in E^n of its frontier is called the *interior of* E^n.

2 Algebra

A *group* G is a collection of elements $g, g', g'', \ldots$ together with a law of composition, written multiplicatively, satisfying the axioms:

(G 1) $(gg')g'' = g(g'g'')$ for all g, g', g'';

(G 2) there exists an element $e \in G$ such that $ge = g$, for all g;

(G 3) to each g there exists $\bar{g} \in G$ such that $g\bar{g} = e$.

The *order* of G is the number (not necessarily finite) of elements in G.

An *abelian group* A is a collection of elements $a, a', a'', \ldots$ together with a law of composition, written additively, satisfying the axioms:

(A 1) $(a+a')+a'' = a+(a'+a'')$ for all a, a', a'';

(A 2) there exists an element $0 \in A$ such that $a+0 = a$, for all a;

(A 3) to each a there exists $(-a) \in A$ such that $a+(-a) = 0$;

(A 4) $a+a' = a'+a$ for all a, a'.

The group G is said to be *commutative* if $gg' = g'g$ for all g, g'. By an abuse of language we may identify a commutative group with the abelian group to which it evidently corresponds. Our main algebraic concern in this book is with abelian groups.

A *subgroup* (more precisely, sub-(abelian group)!) of an abelian group A is a subset of the elements of A which constitutes an abelian group under the law of composition defined in A. The intersection of subgroups is a subgroup.

If B, C are subsets of the abelian group A we write $B+C$ for the subset of A consisting of elements $b+c$, $b \in B$, $c \in C$. We call $B+C$ the *sum* of B and C; this notation evidently extends to any finite collection of subsets of A, or even to an arbitrary collection $\{B_\alpha\}$ provided we select $0 \in B_\alpha$ for all but a finite number of values of α. If each B_α is a subgroup of A, so is their sum.

If A_0 is a subgroup of A, a *coset of* A *by* A_0 is a set of elements $a+A_0$; the *factor group*‡ or *quotient group* A/A_0 is the set of cosets of A by A_0 with the law of composition

$$(a+A_0)+(a'+A_0) = (a+a')+A_0.$$

‡ A strong case exists for calling this the *difference group* and writing it $A-A_0$.

I.2.1 Proposition. *If A_1 is a subgroup of A_0 and A_0 is a subgroup of A, then A_0/A_1 is a subgroup of A/A_1 and*

$$(A/A_1)/(A_0/A_1) \cong A/A_0.$$

I.2.2 Proposition. *If B, C are subgroups of A, $(B+C)/C \cong B/(B \cap C)$.*

A *homomorphism* $\phi : A \to B$ from the abelian group A to the abelian group B is a function satisfying $(a+a')\phi = a\phi + a'\phi$; then $0\phi = 0$ and $(-a)\phi = -a\phi$. The *kernel* of ϕ is the subgroup, $\phi^{-1}(0)$, of A; the *image* of ϕ is the subgroup, $A\phi$, of B; and the *cokernel* of ϕ is the‡ factor group, $B/A\phi$, of B. Then ϕ is a *monomorphism* if its kernel is zero, an *epimorphism* if its cokernel is zero (equivalently, if B is the image of ϕ), and an *isomorphism* if it is monomorphic and epimorphic. We write $0 : A \to B$ for the *zero homomorphism* ($a0 = 0$, all a). An *endomorphism* of A is a homomorphism $A \to A$ and an *automorphism* of A is an isomorphism $A \to A$. We write $\phi : A \cong B$ if ϕ is an isomorphism and we write $1 : A \cong A$ for the *identity automorphism* ($a1 = a$, all a). If A_0 is a subgroup of A, the monomorphism $i : A_0 \to A$, given by $ai = a$, $a \in A_0$, is called the *inclusion* map or *injection*. If $\phi : A \to B$ is a monomorphism then A may be *embedded* in B by identifying a with $a\phi$, all a. If A_0 is a subgroup of A, the epimorphism $p : A \to A/A_0$ which sends each element of A into its coset is called the *projection*.

A sequence of abelian groups and homomorphisms

$$\dots \to A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \to \dots,$$

finite or infinite is *exact at* A_n if the kernel of the homomorphism§ ϕ_n is the image of the homomorphism ϕ_{n+1}. The sequence is *exact* if it is exact at A_n for each n.

I.2.3 Proposition. *The sequence* $0 \to A_0 \xrightarrow{i} A \xrightarrow{p} A/A_0 \to 0$ *is exact.*

The collection of homomorphisms $A \to B$ may be given an abelian group structure by defining the sum $\phi + \phi'$ of homomorphisms ϕ and ϕ' by the rule

$$a(\phi + \phi') = a\phi + a\phi', \quad a \in A, \ \phi, \phi' : A \to B.$$

This group is usually written $\mathrm{Hom}(A, B)$; we shall introduce the notation $A \pitchfork B$, due to E. C. Zeeman.

‡ To complete the 'duality', we may define the *co-image* of ϕ as the factor group $A/\phi^{-1}(0)$.

§ In diagrams it is convenient to write '$A \xrightarrow{\phi} B$' for '$\phi : A \to B$'.

If $\phi: A \to B$, $\psi: B \to C$ are homomorphisms, then the transformation $\phi\psi: A \to C$ is again a homomorphism. If $\phi': A \to B$ and $\psi': B \to C$ are also homomorphisms, then

$$(\phi+\phi')\psi = \phi\psi+\phi'\psi$$
$$\phi(\psi+\psi') = \phi\psi+\phi\psi'.$$

Let $\{A_\alpha\}$ be an indexed family of abelian groups; their *direct sum* is the abelian group A defined as follows: an element of A is a collection of elements $\{a_\alpha\}$, $a_\alpha \in A_\alpha$, subject to the restriction that $a_\alpha = 0$ for all but a finite number of values of α; and addition is defined componentwise by $\{a_\alpha\}+\{a'_\alpha\} = \{a_\alpha+a'_\alpha\}$. We write $A = \sum_\alpha A_\alpha$. If the restriction on $\{a_\alpha\}$ is withdrawn the resulting group is called the *direct product* or *unrestricted direct sum* of the groups A_α and written $\prod_\alpha A_\alpha$. If α runs over a *finite* indexing set, say $\alpha = 1, \ldots, k$, then $\sum_\alpha A_\alpha = \prod_\alpha A_\alpha$ and we write either as $\sum_{\alpha=1}^k A_\alpha$ or $A_1 \oplus \ldots \oplus A_k$. We may embed A_{α_0} in $\sum_\alpha A_\alpha$ (or in $\prod_\alpha A_\alpha$) by identifying a_{α_0} with $\{a_\alpha\}$, where $a_\alpha = 0$, $\alpha \neq \alpha_0$. By a slight abuse of language we refer to the given embedding as an *injection*; similarly, the epimorphism $\sum_\alpha A_\alpha \to A_{\alpha_0}$ $(\Pi A_\alpha \to A_{\alpha_0})$ given by $\{a_\alpha\} \to a_{\alpha_0}$ will be called a *projection*.

The groups A_α are not at the outset subgroups of $\sum_\alpha A_\alpha$. To be precise, we have defined the *external* direct sums of the groups A_α. If the groups A_α are subgroups of an abelian group A such that A is the sum of the groups A_α and each A_α intersects the sum of the remaining A_α in the zero element alone, then A is called the *internal* direct sum of its subgroups A_α.

I.2.4. Proposition. (i) *If A is the internal direct sum of its subgroups A_α, then* $A \cong \sum_\alpha A_\alpha$; (ii) *for any indexed family* $\{A_\alpha\}$, $\sum_\alpha A_\alpha$ *is the internal direct sum of the images under injection of the groups* A_α.

If A_0 is a subgroup of A, and A can be expressed as the internal direct sum of A_0 and some other subgroup of A, then A_0 is said to be a *direct factor* of A.

A set $\{a_i\}$ of elements of an abelian group A *generates* the subgroup A_0 of elements of A expressible as $\Sigma n_i a_i$, the n_i being integers of which only a finite number are non-zero; if the set $\{a_i\}$ is enumerable we may write $A_0 = (a_1, a_2, \ldots)$. If $A_0 = A$ we call $\{a_i\}$ a set of *generators* of A. An identity $\Sigma m_i a_i = 0$ in A is then called a *relation* (between the generators a_i). The *order* of $a \in A$ is the order of the subgroup of A generated by a.

The abelian group F is *free* if it possesses a set of elements $\{f_i\}$ such that each element of F is *uniquely* expressible as $\Sigma n_i f_i$. The set $\{f_i\}$ is called a *basis* for F and the *rank* of F is the number (not necessarily finite) of elements in any basis. This definition is justified by

I.2.5 Proposition. *The rank of F is independent of the choice of basis.*

(This is a classical theorem of linear algebra if the rank is finite. If the rank is infinite it equals the order of F.)

I.2.6 Proposition. *If F is a free abelian group, $\{f_i\}$ a basis for F and A an arbitrary abelian group, then any function defined on the basis with values in A may be uniquely extended to a homomorphism from F to A.*

(For, given such a function θ, we define $(\sum_i n_i f_i)\phi = \sum_i n_i(f_i\theta)$. Then ϕ is single-valued since the representation $\sum_i n_i f_i$ is unique and is clearly homomorphic. Moreover, ϕ extends θ and is the unique such homomorphism.)

A (free abelian) *presentation* of the abelian group A is an epimorphism $\mu : F \to A$, where F is free.

The symbol '$A = J$' means that A is isomorphic with the additive group of integers; the symbol '$A = J_m$' means that A is isomorphic with the residue classes of integers mod m, where m is a positive integer.

Let G be a group. A subgroup G_0 of G is *normal* or *self-conjugate* if $g^{-1}g_0g \in G_0$ for all $g_0 \in G_0$, $g \in G$. If G_0 is a normal subgroup of G, the factor group‡ or quotient group is written G/G_0. The *centre* C of G is the subgroup consisting of elements $c \in G$ such that $cg = gc$ for all $g \in G$.

I.2.7 Proposition. *C is a normal subgroup of G.*

For each $g, g' \in G$ write $[g, g'] = g^{-1}g'^{-1}gg'$ and let $[G, G]$ be the least subgroup of G containing all *commutators* $[g, g']$. Then $[G, G]$ is the *commutator subgroup* or *derived* group of G.

I.2.8 Proposition. *$[G, G]$ is normal in G and $G/[G, G]$ is commutative. Moreover, if G_0 is normal in G and G/G_0 is commutative then $G_0 \supseteq [G, G]$.*

‡ The definitions of subgroup, coset and factor group are analogous to those for abelian groups.