# 1 Introduction and overview

Waves are ubiquitous in nature. They have been studied in the last couple of decades in such diverse forms and varied fields that they may now be said to constitute a new discipline – the science of waves (Lighthill, 1978). This wide and varied interest in waves has been particularly helped by the appearance of that strange entity, the soliton. The wave adopts such diverse forms that it is difficult to present a precise unifying definition. However, we may agree that waves (or disturbances), in an otherwise quiet or uniformly moving medium, have propagation properties and therefore involve the variable time, and have distinct features such as crests and troughs which themselves move with definite speeds. It should, however, be noted that not all waves are oscillatory. Thus, shock waves and solitary waves are not oscillatory. Nevertheless, these are regarded as (nonlinear) entities of great physical importance.

Two major types of waves have been distinguished (Whitham, 1974). The first is called hyperbolic and requires the system of $n$ governing partial differential equations to have $n$ real characteristic directions and correspondingly $n$ linearly independent left eigenvectors of the relevant matrix (Courant & Hilbert, 1962). The second type of waves, called dispersive, are categorised by a real dispersion relation connecting the frequency and wave number (Bhatnagar, 1979). These definitions are broadened suitably to apply to partial differential equations with variable coefficients as well as nonlinear ones.

There is another type of nonlinear wave which is diffusive and which is epitomised by the equation

$$u_t + uu_x = \frac{\delta}{2}u_{xx}. \tag{1.1}$$

This is the celebrated Burgers equation. Here $\delta$ is a (small) coefficient of viscous diffusion. This is a nonlinear parabolic equation and describes in a simple manner a balance between nonlinear convection and linear diffusion or dissipation. This equation and its generalisations – scalar as well as vector – describe phenomena in such a variety of situations that they deserve a distinct categorisation, namely nonlinear diffusive wave equations. It must be recognised that, in the limit of $\delta \to 0$, eq. (1.1) goes into a scalar hyperbolic equation. Indeed, eq. (1.1) was suggested as a model to describe the structure of shock waves in gas dynamics, which is missed by the hyperbolic type of equations for which the shock appears as a sharp discontinuity.

A considerable part of the present monograph is devoted to the discussion of the Burgers equation and its generalisations (GBEs). However, it must be emphasised that the Burgers equation by no means exhausts the nonlinear diffusive phenomena. To stress this point and to bring out the contrast between nonlinear convective diffusive equations and those without convection, we discuss in detail several other nonlinear model equations in chapter 4. These include Fisher's equation, a nonlinear heat equation etc. For most of this monograph, we consider scalar equations only.

Chapter 2 begins with a heuristic derivation of the Burgers equation. This is followed by an order-of-magnitude analysis of the Navier–Stokes equations to derive a coupled system of two equations describing one-dimensional waves of finite amplitude in a viscous and heat-conducting gas, a generalised Burgers system. This system, under further approximation, delivers the Burgers equation. After a brief historical account of this equation, and the Hopf–Cole transformation which exactly linearises it to the heat equation, a pure initial value problem is posed and solved in a simple manner via the corresponding problem for the heat equation. Special solutions describing important physical situations such as the travelling shock wave, the single hump, the $N$ wave and the periodic profile are derived. Continuous or distribution functions as initial conditions are assumed, and physical (or dimensional) arguments pointing to the similarity form of the solutions for travelling shock and single hump are described. These special solutions are carefully analysed in each of the several temporal and spatial domains that arise from considerations of the importance and balance of different terms. This is motivated by the desire to find analytic solutions, at least in some of the domains, of GBEs in subsequent chapters, for which no Hopf–Cole-like transformation exists. Although most of the earlier investigations relate to initial value problems, we also pose, in the semi-infinite domain, a boundary value problem for the Burgers equation and use a certain equivalence theorem for the heat equation, between initial value problem over the whole real line and a boundary value problem over the positive real line, to recover earlier solutions now arising from certain boundary conditions.

The Burgers equation is very important from the mathematical point of view as a canonical form since it highlights clearly the nature of analytic solutions in various temporal and spatial domains, which become available due to the Hopf–Cole transformation. The equations that arise in physical applications are more general than the Burgers equation and do not, in general, admit exact analytic solutions. Chapter 3 treats GBEs. After a brief review of the singular perturbation methods, we employ them to find a uniformly valid solution to order $\delta$ for the GBE, which, besides usual terms, has a linear damping as an additional term. We then derive from the Navier–Stokes equations a model which combines the effect of spherical or cylindrical expansion besides nonlinearity, viscous diffusion and heat conduction. This is achieved by using the method of multiple scaling. The nonplanar GBEs are studied analytically, as far as possible, using matched asymptotic expansions, in certain of the temporal domains for the sharp $N$ wave initial profile. Reference is made to the gaps which still remain unbridged. A kindred discussion relates to the solution of the harmonic boundary value problem arising from a piston motion. For this purpose, it makes more sense to pose a boundary value problem in a semi-infinite domain, altering in the process the basic equation so that the roles of distance and (retarded) time are interchanged. Significant physical and mathematical consequences of the solution of the harmonic problem are carefully analysed. Here, even though we treat the standard Burgers equation, we include it in the chapter on generalised Burgers equations because of the complexity introduced by the boundary conditions. Generalised Hopf–Cole transformations are used to find a whole class of GBEs which may be changed into linear parabolic equations with variable coefficients. Some of the physically relevant equations are identified and their special solutions are discussed. Prominent among these is an inhomogeneous Burgers equation which occurs in several physical contexts.

While a major part of this monograph is concerned with equations of Burgers type which have a convective term as an important element responsible for wave steepening and shock phenomena, chapter 4 deals with a few representative nonlinear diffusion equations wherein the convective term is absent. This has been done for two reasons: firstly, to visualise how other nonlinear diffusion phenomena compare with that simulated by the equations of Burgers type, and, secondly, to prepare the ground for the discussion of stability (or intermediate asymptotic) analysis for a variety of nonlinear diffusion equations. The equations in this chapter include Fisher's equation, a nonlinear heat equation and an equation from plasma physics which has a (spatially) variable coefficient besides nonlinearity. While for the former equations the special solutions we study belong to the usual similarity form, the latter have a product form. Indeed, the reader will notice a strong undercurrent of the similarity approach in the entire course of the present monograph. This is partly due to the author's bias and partly due to the conviction that the similarity analysis leads to some of the most bona fide exact solutions of nonlinear problems. Historically, the similarity/product solutions were viewed as special solutions of nonlinear partial differential equations, their chief distinction being that they were governed by nonlinear ordinary differential equations which could be solved more conveniently either in a closed form or numerically. This viewpoint has since undergone a change. These special solutions represent what are referred to as intermediate asymptotics, 'describing the behaviour of the solutions to the original equations for a wide class of initial, boundary and mixed problems, away from the boundaries of the region of independent variables or, alternatively, in a region where in a sense the solution is no longer dependent on the details of the initial and/or boundary conditions but is still far from being in a state of equilibrium' (Barenblatt and Zel′dovich, 1972). In chapter 4, we discuss the role of the similarity/product type of solutions as intermediate asymptotics for a few representative nonlinear diffusion equations. That is to say we study the solutions of a class of initial and/or boundary value problems for these equations which evolve into self-similar solutions, as well as the manner and mode of such evolution. The equations that we treat here include a nonlinear heat equation, a nonlinear diffusion equation of plasma physics with a variable coefficient and the GBEs in spherical and cylindrical symmetry.

The analytical studies reported in chapter 3 clearly bring out the gaps in the understanding of the solutions of the GBEs. For example, for the non-planar GBE, there are several domains – the embryonic shock region and the infinitely long (in time) one beyond the Taylor-shock region – for which the analytical form of the solution seems difficult to obtain. Indeed, even the final phase of the $N$ wave propagation, which is essentially linear, remains undetermined to the extent of an unknown multiplication factor for the cylindrically symmetric case. Therefore, there is a need to have a thorough understanding of 'good' numerical solutions of these equations which might, in turn, suggest the analytic form of the solution. In the final chapter of this monograph, we discuss two numerical techniques – implicit finite difference and the pseudo-spectral (accurate space differencing) – for three nonlinear diffusion equations, namely Fisher's equation, the GBEs in spherical and cylindrical symmetries and the GBE with a damping term. The need for using the pseudo-spectral approach becomes imperative for discontinuous initial data which the implicit scheme is not able to handle in an effective and accurate manner and which the pseudo-spectral scheme is. However, once the discontinuous profile has smoothed out and has settled down, say, to one with a Taylor shock, the implicit difference scheme can take over and deliver accurate results with great economy, in comparison with the pseudo-spectral approach which, though very accurate, is expensive in terms of computer time. The numerical techniques help understand the intermediate asymptotic nature of the travelling wave solution of Fisher's equation, the decay of spherical and cylindrical $N$ waves and of the single hump initial profile evolving under a GBE with a damping term or under non-planar GBEs.

The present monograph is almost entirely devoted to scalar diffusive equations. Reference may be made to Smoller (1983) for systems of equations describing, in particular, reaction-diffusion processes. Moreover, we have restricted ourselves mostly to the gas dynamic context of the Burgers equation and its generalisations. For applications to turbulent flows, we refer the reader to Burgers (1974), Gurbatov *et al.* (1983) and Qian (1984). The review article of Gurbatov *et al.* contains a large bibliography.

While the major applications of the nonlinear diffusive equations have been drawn from gas dynamics, it will become apparent from the references that they occur frequently in many other areas such as plasma physics, heat conduction, elasticity, biomathematics etc. Therefore, the material in this monograph should be useful to scientists and engineers working in these areas. The treatment of the problems in the monograph is mainly applied mathematical in nature; however, the physical explanation is also briefly provided.

The prerequisites for the present monograph are a basic course in gas dynamics, and a fair knowledge of ordinary and partial differential equations. In particular, familiarity with the theory of parabolic partial differential equations will be found helpful.

# 2 The Burgers equation

## 2.1 Introduction

Wave phenomena are, in general, governed by nonlinear systems of partial differential equations subject to certain physically motivated initial and/or boundary conditions. The Navier–Stokes equations represent a typical example of such a system. These systems, in most cases, cannot be solved by exact analytic approaches. Indeed, even the numerical solution of these systems poses severe difficulties. Thus, in recent years, there have been attempts to derive simpler equations using perturbation methods, which retain from the larger systems the essentials of the physical problems and which hold over extensive spatial and temporal domains. The fact that these model equations commonly appear in a variety of physical contexts attests to their importance. Furthermore, recent investigations have shown that various model equations governing similar physical phenomena enjoy unifying mathematical properties. The best known examples are the Burgers equation

$$u_t + uu_x = \frac{\delta}{2} u_{xx}, \tag{2.1}$$

and the Korteweg–deVries equation

$$u_t + \sigma uu_x + u_{xxx} = 0. \tag{2.2}$$

While our main concern will be with eq. (2.1), we shall often compare and contrast eqs. (2.1) and (2.2) and their solutions, since the study of these *apparently* similar equations has provided mutual enrichment and led to important results for kindred classes of equations.

We commence our discussion with the system describing plane compressible flows in an ideal (polytropic) gas ignoring dissipative effects, viz.

$$\rho_t + v\rho_x + \rho v_x = 0, \tag{2.3}$$

$$\rho(v_t + vv_x) + p_x = 0, \tag{2.4}$$

$$p = k\rho^\gamma, \quad S = \text{constant}. \tag{2.5}$$

Here $\rho, v$ and $p$ are the density, particle velocity and pressure, respectively, depending on the spatial co-ordinate $x$ and time $t$. $S$ is entropy, assumed to be constant, $\gamma = c_p/c_v$, the ratio of specific heats, and $k$ is a constant. If we consider a small disturbance over a uniform quiescent medium ($u = 0$, $\rho = \rho_0, p = p_0$), we may linearise the system (2.3)–(2.5) and obtain an equation, by suitable elimination, describing the disturbances in any of the variables

$$v' = v, \quad p' = p - p_0, \quad \rho' = \rho - \rho_0: \tag{2.6}$$

$$\rho'_{tt} - a_0^2 \rho'_{xx} = 0, \tag{2.7}$$

$$a_0^2 = a^2(\rho_0, S_0), \quad a \text{ being the speed of sound}$$

(see Courant and Friedrichs (1948, p. 19)). Eq. (2.7) is the 'standard' wave equation which has the general solution

$$\rho' = \phi(x - a_0 t) + \psi(x + a_0 t) \tag{2.8}$$

where $\phi$ and $\psi$ are arbitrary appropriately differentiable functions of their arguments. The relation (2.8) describes the solution $\rho$ of any initial value problem for eq. (2.7), the functions $\phi$ and $\psi$ being determined by the initial conditions. These functions describe waves splitting from the initial conditions and moving to the right and the left, respectively, with speed $a_0$. We shall subsequently refer to solutions depending on either $x - a_0 t$ or $x + a_0 t$ only, as travelling or stationary waves. Now, if we restrict ourselves to waves moving to the right so that $\psi(x + a_0 t) \equiv 0$, the function $\phi(x - a_0 t)$ satisfies a component of eq. (2.7), namely

$$\rho'_t + a_0 \rho'_x = 0. \tag{2.9}$$

(Of course, it satisfies eq. (2.7) too). Eq. (2.9) is the simplest linear wave equation with solution $\phi(x - a_0 t)$ assuming the initial value $\phi(x)$ and giving at later time the same profile translated as a whole to the right a distance $a_0 t$, without any change in form.

Eq. (2.7) was derived on the assumption that the perturbations in pressure etc. were infinitesimally small. This is not generally true if the agency producing the wave releases large energy or momentum. This is the case, for example, with an explosion, or a relatively fast piston motion. The system (2.3)–(2.5) in essentially two variables ($\rho$ and $v$, say) is nonlinear and difficult to handle in complete generality. Nevertheless there has been considerable analytic interest in this system. In particular, some progress can be made by seeking simple wave solutions such that one of the dependent variables is a function of the other. This procedure is due originally to Earnshaw (1858). Rewriting eqs. (2.3)–(2.4) in terms of $v$ and $\rho$

only, by introducing the square of the speed of sound, $a^2 = (\partial p/\partial\rho)_{S=S_0} = k\gamma\rho^{\gamma-1}$, where $k$ is a constant, we have

$$\rho_t + v\rho_x + \rho v_x = 0, \tag{2.10}$$

$$v_t + vv_x + \frac{a^2(\rho)}{\rho}\rho_x = 0. \tag{2.11}$$

Now, we assume that $v = V(\rho)$ so that (2.10)–(2.11) become

$$\rho_t + (V + \rho V')\rho_x = 0, \tag{2.12}$$

$$\rho_t + \left(V + \frac{a^2}{\rho V'}\right)\rho_x = 0. \tag{2.13}$$

Here a prime denotes differentiation with respect to $\rho$. This system of linear algebraic equations in $\rho_t$ and $\rho_x$ has a non-trivial solution provided the determinant of the coefficient matrix vanishes so that

$$V' = \pm\frac{a}{\rho} = \pm a_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}\frac{1}{\rho}. \tag{2.14}$$

The system (2.10)–(2.11) then reduces to one of the equations

$$\rho_t + (V \pm a)\rho_x = 0, \tag{2.15}$$

where

$$V(\rho) = \int_{\rho_0}^{\rho}\frac{a(\rho)}{\rho}\mathrm{d}\rho = \frac{2}{\gamma-1}\{a(\rho) - a_0\}. \tag{2.16}$$

Restricting attention to waves moving to the right and choosing, therefore, the plus sign in eq. (2.15), we now write the corresponding equation for $v$. This follows easily from multiplying eq. (2.12) or eq. (2.13) by $V'(\rho)$ and writing the result in terms of $v$ via eq. (2.14). Thus, we obtain

$$v_t + \left(a_0 + \frac{\gamma+1}{2}v\right)v_x = 0. \tag{2.17}$$

The 'simplicity' of the simple wave solution given by eqs. (2.15)–(2.16) or eqs. (2.16)–(2.17) does not arise from any meddling with nonlinearity; these equations are typically nonlinear. The mathematical problem has been reduced to solving an initial value problem for the single first order nonlinear partial differential equation (2.17) consistent with the intermediate integral (2.16) relating $v$ and $a(\rho)$, and hence $v$ and $\rho$. We note in passing that this argument has been extended to an $n$th order system of homogeneous PDEs by Schindler (1970). (See also Levine (1972) and Rott (1978).)

If we compare eq. (2.9) with eq. (2.17), we observe that the propagation speed is $a_0$, a constant, for the former, while it is $a_0 + \frac{1}{2}(\gamma + 1)v$, a function of the dependent variable $v$ itself, for the latter. If an arbitrary profile, however smooth, is chosen initially for $v$, it is well known and can be graphically checked by drawing the characteristics in the $x$–$t$ plane that the solution of eq. (2.17), after a certain time, ceases to be single valued. The parts of the initial profile with higher values of $v$ travel faster than those with lower values so that, in due course, in the compressive parts of the initial profile we have three values of the solution, which is impossible. Physically, what transpires is that a shock is formed at a point of the profile where the gradients are large. In a thin neighbourhood of this point, due to the prevalence of large gradients, irreversible thermodynamic processes such as viscosity and heat conduction which were ignored in the derivation of eq. (2.17) intervene. The steepening gradients are eased and a certain balance is struck. The shock with a 'small' thickness then heads the smooth parts of the profile. The details of the shock formation and its subsequent decay in the framework of (the non-viscous and non-heat-conducting equation) (2.17) may be found in Whitham (1974).

Thus, the model equation (2.17) is inadequate to describe flows with shocks and therefore it must be improved upon to include the neglected effects of viscosity and heat conduction. (This was indeed the way it was done in the early stages of the evolution of the topic.) In a heuristic way, eq. (2.17) was 'embedded' with viscosity, so that we obtain

$$v_t + \left( a_0 + \frac{\gamma + 1}{2} v \right) v_x = \frac{\delta}{2} v_{xx} \tag{2.18}$$

(see Cole (1951)). Here $\delta$ is a small parameter. This equation can be transformed into eq. (2.1) by a simple change of variables. Eq. (2.1) represents a simple $(1 + 1)$-dimensional model, combining a nonlinear convective term and a small linear viscous term.

## 2.2 Derivation of Burgers' equation

The Burgers equation and its generalised forms relevant to various physical circumstances have been derived by several investigators using perturbation methods and multiple scales. Here we follow an order-of-magnitude argument due to Lighthill (1956), which leads in the process also to a coupled system of two equations intermediate between the Navier–Stokes equations and the Burgers equation. This system has some intrinsic interest, since its left hand sides are exactly those of one-dimensional isentropic gas dynamic equations (rather than the simple wave form as in the left side of eq. (2.18)), and the right side in one of the equations contains linearised viscous and heat-conduction terms. Thus, starting with the plane Navier–Stokes equations and appropriately grouping various terms (in a certain fashion) we have

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x} + \left( v \frac{\partial \rho}{\partial x} \right) = 0, \tag{2.19}$$

$$\frac{\partial v}{\partial t} + \left( v \frac{\partial v}{\partial x} \right) + \frac{1}{\rho} \frac{\partial p}{\partial x} = \left[ \frac{\frac{4}{3}\mu_0 + \mu_{v0}}{\rho_0} \frac{\partial^2 v}{\partial x^2} \right]$$
$$+ \left\{ \frac{1}{\rho} \frac{\partial}{\partial x} \left( \left( \frac{4}{3}\mu + \mu_v \right) \frac{\partial v}{\partial x} \right) - \frac{\frac{4}{3}\mu_0 + \mu_{v0}}{\rho_0} \frac{\partial^2 v}{\partial x^2} \right\}, \tag{2.20}$$

$$\frac{1}{\gamma - 1} \frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t}$$
$$= \left\{ \left( \frac{4}{3}\mu + \mu_v \right) \left( \frac{\partial v}{\partial x} \right)^2 \right\} + \left[ k_0 \frac{\partial^2 T}{\partial x^2} \right] + \left\{ \frac{\partial}{\partial x} \left( \frac{k \partial T}{\partial x} \right) - k_0 \frac{\partial^2 T}{\partial x^2} \right\},$$

$$\mathrm{D} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}. \tag{2.21}$$

Here $\mu$ is the viscosity coefficient equal to the ratio of shear stress to rate of shear, $\mu_v$ is the bulk viscosity, and $k$ is the thermal conductivity. $T$ stands for temperature. The suffix 0 denotes values in the undisturbed condition. The constant coefficients $k_0$, $\mu_0$ and $\mu_{v_0}$ are known to be small. The above grouping needs some explanation. The unbracketed terms are the largest and lead to the linearised equation (2.7). If $V_0$ and $a_0$ denote a characteristic particle velocity in the wave and the undisturbed value of the speed of sound respectively, then the terms in round brackets are of the order of $V_0/a_0$ and those in square brackets are of the order $\nu\omega/a_0^2$ as compared to the unbracketed ones, respectively. The terms in curly brackets are perturbations over the linearised form of viscosity and heat conduction and therefore smaller than those without brackets by an order $(\nu\omega/a_0^2)\,(V_0/a_0)$. To verify these statements, we may use the linear solution

$$\frac{v}{a_0} = \frac{\rho'}{\rho_0} = \frac{p'}{p_0} = \phi\left( \omega \left( t - \frac{x}{a_0} \right) \right), \quad \text{say,} \tag{2.22}$$