DIMENSION THEORY OF GENERAL SPACES

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INTRODUCTION

This book is concerned with the dimension theory of general topological spaces. It provides a complete and self-contained account of the theory laying particular emphasis on the dimensional properties of non-metrizable spaces. It is intended to serve as a reference work for mathematicians with an interest in general topology. It is written in such a way as to be accessible to beginning postgraduate students and might be used as a textbook.

A theory of dimension starts with a 'dimension function', which is a function d defined on the class of topological spaces such that $d(X)$ is an integer or ∞, with the properties that $d(X) = d(Y)$ if X and Y are homeomorphic and $d(\mathbf{R}^n) = n$ for each positive integer n, where $\mathbf{R}^n$ denotes Euclidean n-space. Dimension theory reveals the properties of dimension functions. For example we wish to know under what circumstances it can be asserted that $d(A) \leqslant d(X)$ if A is a subspace of a topological space X. Assertions of this form are known as subset theorems for the dimension function d. An affirmation that $d(X) = \sup d(A_\lambda)$ for certain coverings $\{A_\lambda\}$ of some spaces X is called a sum theorem for the dimension function d. In dimension theory we also examine the relationships between different dimension functions. An objective is to ascertain circumstances under which 'reasonable' dimension functions coincide.

The principal dimension functions are the covering dimension function which is studied in Chapter 3 of this book and the small and large inductive dimension functions which are studied in Chapter 4. Chapters 5 and 6 are devoted to a study of related dimension functions. There is a substantial theory of covering dimension for normal spaces and the covering dimension function satisfies a general subset theorem for the class of totally normal spaces. The large inductive dimension function satisfies sum and subset theorems for totally normal spaces. The small inductive dimension function has the greatest intuitive appeal and satisfies the subset theorem for arbitrary spaces. There is no theory of small inductive dimension but, being easier to calculate, the small inductive dimension function gives information about the other dimension functions. The coincidence of all three principal dimension functions for separable metrizable spaces explains the elegance of classical dimension theory, so well exposed by Hurewicz and Wallman [1941]. Chapter 7 shows that a very satisfactory theory of dimension can be constructed for metrizable spaces, but also includes P. Roy's example of a metrizable space for which small inductive dimension is different from covering dimension and large inductive dimension. Most of Chapter 8 is occupied with the pathological dimension theory of bicompact spaces and includes an example due to V. V. Filippov which shows that the small and large inductive dimensions can differ for such spaces. There are many other examples throughout the book. In Chapter 10 the covering dimension function is modified for non-normal spaces. Dimension-theoretic applications of the algebra of continuous bounded real-valued functions on a topological space are made in this chapter and Katětov's beautiful algebraic characterization of the dimension of a Tihonov space is given. Information about the dimension of bicompactifications is obtained here and in Chapter 6. Universal spaces for various classes of spaces of given dimension are constructed during the course of the work, which concludes with Pasynkov's universal spaces for Tihonov spaces of given dimension.

In 1968 I gave postgraduate lectures on dimension theory at the University of London. The notes of those lectures provided a starting point for the writing of this book. My objective was to write an account of the now mature theory of dimension within point–set topology. I hope this book will be read by postgraduate students at the beginning of their careers. The reader is assumed to be familiar with naive set theory and its standard notation and to know as much general topology as one might expect an undergraduate course to cover. The first section of the first chapter contains a brief résumé of elementary topology. The rest of Chapter 1 is concerned with standard general topology and will be used for reference only by many readers. Chapter 2 covers material on paracompactness and metrization which may be less familiar. Both chapters contain some topics which arose in dimension theory and these may be unfamiliar to readers well acquainted with general topology. Expanded treatments of many of the topics in Chapters 1 and 2, and the examples necessary to complement the theory, will be found in the books of Dugundji [1966], Engelking [1968] and Kelley [1955]. I am glad to acknowledge my debt to these works.

Throughout the book proofs are given in detail except for those of some standard results on normality and compactness in Chapter 1. The symbol * follows the statement of these propositions, with which it is assumed the reader will be familiar. Their proofs are given in many textbooks. The Brouwer theorem, that the unit sphere in Euclidean space is not a retract of the closed unit ball which it bounds, is used in the determination of the dimension of Euclidean space. An *ad hoc* proof of this theorem might have been given but it seems that the 'right' proof is by means of homology theory. I hope the omission of its proof will be acceptable since its truth is easy to take on trust. An effort has been made to employ separation properties of topological spaces only where they are appropriate. In particular normal and paracompact spaces are not assumed to satisfy the Hausdorff separation axiom. In places where a known theorem and its proof have been modified in accordance with this principle, then the new proof usually seems to be more natural.

The book is written in the 'definition–proposition–proof–remark' style. I hope this will allow the logical structure of the subject to show clearly. Much of the informal and motivational material has been collected into notes at the end of each chapter. Each chapter is divided into sections. The items (definitions, theorems, propositions, corollaries, examples, remarks) in a section are numbered by pairs $n.p$ of integers, where n is the section number within the chapter and p is the item number within the section. Elsewhere in the book, item $n.p$ of Chapter m is referred to as item $m.n.p$ except in Chapter m itself where the reference is abbreviated to $n.p$. A Halmos symbol □ marks the end of a proof. The appearance of □ immediately after the statement of a proposition or corollary signifies that its truth is obvious or occasionally that its easy proof is left to the reader.

There are notes at the end of every chapter except Chapter 1. These contain the references to original sources, listed in the bibliography. There are also comments on the historical development of the subject. These do not form a complete record and there has been no attempt to make a definitive assignment of credit. I hope some impression is given of the subject as a human activity, which will add to the reader's enjoyment. The notes also survey some recent developments which have not been included in the book. I made much use of the survey articles of Aleksandrov [1951, 1960, 1964] and Nagata [1966, 1971] whilst writing the notes.

I have been helped by several people whom I wish to thank. Professor C. H. Dowker of Birkbeck College, London, helped me to plan the course of lectures from which the book has grown. He has answered many questions and advised me most generously as the work has developed. I was sustained during the labour of transforming

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