Introduction

The idea of symmetry is familiar in everyday life, whether applied to solid objects or to patterns and designs. The essential feature of a symmetrical object is that it can be divided into two or more identical parts: and furthermore that these parts are systematically disposed in relation to one another. In addition some objects, such as a ladder or a wallpaper pattern, have repetitive elements, whereas others, for example a table or a handcart, though clearly symmetrical, are not repetitive.

One might well ask whether there can exist an unlimited variety of symmetrical patterns and objects or whether it is possible to classify them, grouping them according to the characteristics they have in common. The aim of this book is to show how such a classification can be undertaken, and indeed to go a step further by enumerating all the possible types of symmetry, both for patterns and solid objects, either with or without repetitive elements.

We shall find that, while a single object may exhibit any one of an infinite range of symmetry types, there are severe limitations on the number of types of repetitive pattern. Thus it will appear that there are only 7 types of 'frieze pattern', 17 of 'wallpaper pattern', and so on. Many varieties of symmetry are found in decorative art and in the natural and other forms that surround us. But some natural forms, such as those of shells and plants, exhibit a structure in which parts are similar but differ in size, often diminishing at a uniform rate. Such forms, which are characteristic of many growth patterns, are mentioned briefly in Chapter 10.

The two main kinds of symmetry found in plane patterns are based on reflexion and rotation. That is, one part of the pattern can be brought into coincidence with another part by one of these two means. Examples of symmetry by reflexion are the letter A and the Latin cross: patterns having rotational symmetry are the letter Z and the swastika. It is also possible for a pattern to exhibit both kinds of symmetry together, as in the case of the Greek cross or the letter I. With repetitive plane patterns, such as that of a parquet floor, we shall find that there is a third kind, called *glide reflexion* symmetry (Fig. 0.01).

Fig. 0.01

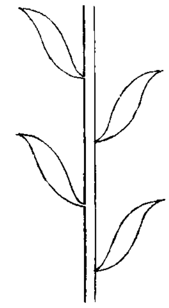
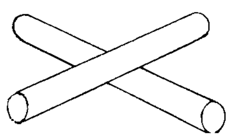


Fig. 0.02



Solid objects too can have reflexional and rotational symmetry and will often have both. We shall show that in addition there are several other kinds of symmetry in three dimensions. First there is *inversion* symmetry, as exhibited by the pair of cranks of a bicycle; such symmetry may be found in conjunction with one or both of the other types. Next there is *rotatory inversion*. A simple example of this is provided by two equal sticks laid across each other at right angles (Fig. 0.02). If the cross were turned through a right angle and then inverted, the sticks would have changed places. Finally, repetitive three-dimensional patterns may have *screw symmetry*, an example of which is seen in

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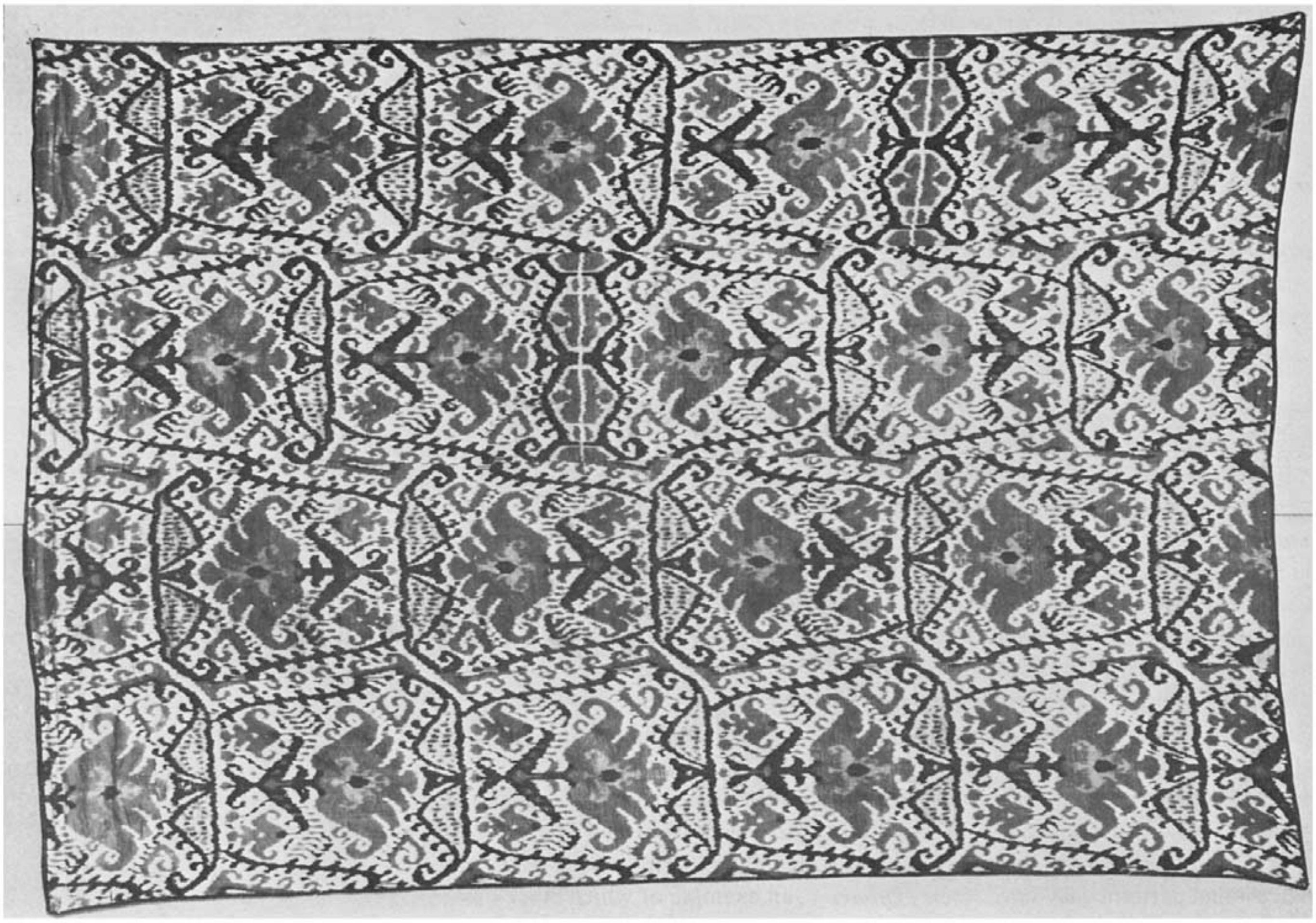
a 'spiral' staircase. As a result of the existence of all these possible symmetries it is not surprising that the symmetry types for three dimensions are much more numerous than those for two.

The physical process of growth, be it of crystals or of living matter, favours the production of symmetrical forms; and indeed the world is pervaded by shapes having a greater or lesser degree of symmetry, from the leaf to the human form. For this reason there must be a sense of symmetry in the mind of every artist. In the applied arts, from the simplest hand-made pot to the Palladian country house or the Gothic cathedral, symmetry appears very often as a prominent feature of the design; and when it does not, there has usually been a deliberate and conscious effort to avoid it. In the design shown in Fig. 0.03 the carefully contrived avoidance of symmetry (in the pattern taken as a whole) is so obvious as to make one realise that symmetry and repetition come naturally: it is the avoidance of them that is artificial.

It is in Arabic and Moorish design that the different kinds of symmetry have been most fully explored. The later Islamic artists were forbidden by their religion to represent human or even animal form, so they turned naturally to geometrical elaboration. Among their works, and also among those of ancient Egypt, numerous examples are to be found of all the 7 types of 'frieze' pattern and most, if not all, of the 17 possible types of 'wallpaper' pattern.

Fig. 0.03



Friezes and wallpapers are examples of two-dimensional patterns that are repeated indefinitely in one, or two, directions respectively. Their possible types can be described mathematically in terms of two-dimensional *line groups* and *plane groups*. These are groups of movements that bring the whole pattern into self-coincidence. There are also two-dimensional patterns that are not repeated and they are described by means of *point groups*, that is, groups of movements that leave one point fixed. Patterns having various types of point symmetry include letters of the alphabet, many commercial symbols, the patterns produced by the kaleidoscope, and the rose windows of cathedrals. As well as in friezes, two-dimensional line symmetry is found in the decorative borders of oriental rugs and in the edges of mosaic pavements. Similarly, two-dimensional plane groups describe the decorative patterns applied to surfaces, as seen in wallpapers, tessellated pavements, mosaics and tiled arabesques, and the repetitive patterns of many carpets and rugs. Needlework, too, offers many examples. Fig. 0.04 shows a mid-nineteenth-century English sampler in which many types of frieze and wallpaper pattern can be seen.

Three-dimensional point groups can be used to classify the many types of solid object having point symmetry. Lampshades and chandeliers can have very interesting symmetry properties, as also have many items of jewellery, such as bracelets and brooches. Of natural forms, the 32 classes into which crystals are grouped according to their external shapes correspond to the 32 'crystallographic' point groups. Among the most familiar of these are the cubic crystals of common salt and the six-pointed crystals of snow.

Three-dimensional line symmetry is that possessed by objects having repetition in one direction only, such as ropes, chains, necklaces and plaits. Line symmetry is also found in carved or decorated columns and straight staircases and balustrades, and in the structure of some buildings, particularly in arcades and cloisters. It is a feature too of many important chemical molecules such as those of proteins and polymers and of DNA, the basis of life. The growth pattern of many plants shows approximate line symmetry along the stem.

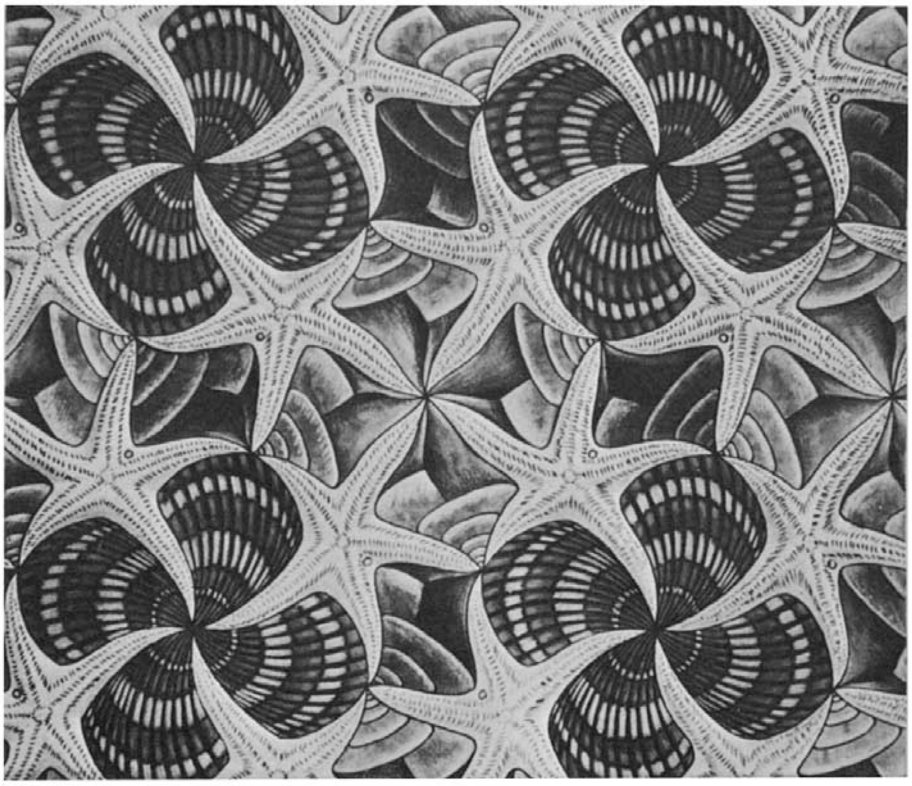
Fig. 0.04

Plane symmetry in three dimensions is the property of almost all fabrics, including woven and knitted materials, also of carpets and rugs, basket work and cane work. In the realm of architecture we find that the stone screens of Moslem buildings, as well as brick walls and roof tiling, exhibit various types of plane symmetry.

Finally, in three dimensions, it is possible to have patterns repeated indefinitely in all three directions, or throughout space. Such patterns are described by *space groups*, of which there are no less than 230. Few of them are met with in ordinary experience; however, they are fundamental to matter in the solid, since any crystalline material has an internal structure based on one of these groups. (It has been shown during the present century, by means of X-rays, that the atoms in a crystal are always arranged in a regular formation belonging to one of the 230 groups. If they were not so arranged the material would be not crystalline but amorphous, like glass.) The nearest approach to space symmetry in everyday life is in the various methods available for packing similar objects, such as tennis balls, into a box, or of piling bricks into a stack. The mathematically minded can also consider the symmetry of space tessellations, that is, the various ways of filling space with one or more kinds of identical solids.

In the free arts of painting and sculpture perfect symmetry is more often avoided than pursued, being replaced usually by a sense of balance, an equality of 'weight' rather than of measurement. However, some modern artists, notably M.C. Escher and M. Vasarely, have used various degrees of plane and point symmetry to very fine effect. Fig. 0.05 shows one of M.C. Escher's designs and two more are illustrated in Figs. 11.14 and 11.15.

Fig. 0.05



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One of the fascinations of the study of symmetry lies in the relationships revealed between a wide variety of objects, natural and artistic. A snow crystal, the bee's cell, a patchwork quilt and the basalt rocks of the Giant's Causeway have something in common, as represented by the word 'hexagonal', yet their forms differ in certain important respects. The mathematical concept of symmetry provides a classification for such similarities and differences and can thus be used as a common basis of description for much of the visual world.

Part I Descriptive

To describe the symmetry of a pattern or object one must consider its division into identical, or congruent, parts and the way in which those parts are related positionally to one another. It is natural and convenient to describe such relationships in terms of movements. Arrows pointing north, south, east and west suggest the movements of a weathercock, and the arms of the Isle of Man suggest rotations through 120°. A repeating pattern, such as a frieze, suggests translatory movements, the movements of a procession.

All these are physical movements, but a pair of wrought iron gates suggests a reflexion, which is not exactly a movement, unless we count Alice's movement through the looking-glass. We shall, however, find it convenient to give the word *movement* this slightly extended meaning. A left hand and a right hand can be placed so that each is a reflected image of the other and we shall call the change from one to the other a 'movement', although it is not possible for the right hand to be moved physically into the space the left hand was occupying.

These movements, and some combinations of them, enable us to describe in the following chapters the various kinds of symmetry. But the movements also serve another purpose. They are the means by which complicated symmetrical patterns can be built up from simple elements. This is the principle of the kaleidoscope, which uses two reflexions to produce patterns containing rotations and further reflexions as well. Thus two movements in combination imply others.

Such relationships between movements will form a main part of our study. They determine what is at choice and what follows automatically. This is a question of interest to the designer and it will receive some attention in Part I. It is also the kernel of the mathematical approach, as will be apparent to the reader of Part II.

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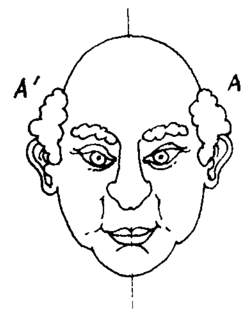
1

Reflexions and rotations

The front view of the human face, as it appears in a drawing, is a good, if not perfect, example of *symmetry about a line.* Every point A on one side of the line has its counterpart A' on the other side (Fig. 1.01) such that AA' is bisected at right angles by the line. It is natural to think of the line as a mirror and of A and A' as mirror images of each other. We shall call the line a *mirror line* or *reflexion line* and we shall use the word reflexion to mean the change from one side of the figure to the other.

Fig. 1.01



This is in two dimensions: but the human face is really three-dimensional and a mirror is really a plane, so in three dimensions we have a *mirror plane* rather than a mirror line. The 'movement' which we call *reflexion* changes A to A' and A' to A, thus leaving the whole figure self-coincident or, as we may conveniently say, unchanged.

The two sides of the figure are *congruent*. This word means that two figures, or parts of a figure, are so related that for every point of one there is a corresponding point of the other and that the distance between any two points of the one is equal to that between the corresponding points of the other. It follows that corresponding angles are equal, but it is to be noted that, in the case of reflexion, the turns represented by corresponding angles are in opposite senses, clockwise and anticlockwise (Fig. 1.02); and in three dimensions corresponding turns are right-handed and left-handed (Fig. 1.03). The two parts of the figure are said to be *indirectly* or *oppositely congruent*.

Fig. 1.02

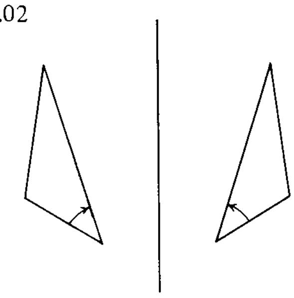
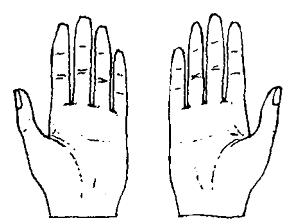
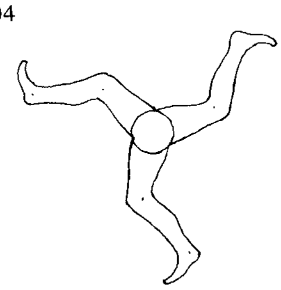


Fig. 1.03



Rotation suggests another kind of symmetry (Fig. 1.04) and here the congruence is *direct*, each part of the figure being an exact reproduction of another part, though differently placed. Rotation is a *proper* or *direct movement.* Here again it should be noted that the turn, applied to the figure as a whole, leaves it unchanged.

Fig. 1.04



A movement or transformation that changes a figure into a congruent figure is called an *isometry*. Rotation is a *direct isometry* and reflexion is an *indirect* or *opposite isometry. Symmetry* means that the parts of a figure are not only congruent* but related by an isometry, e.g. reflexion or rotation, in such a way that the whole figure is self-coincident under that isometry. A *symmetry movement* is one that changes every part of a figure into another part, leaving the figure as a whole unchanged. Obviously it may be repeated any number of times with the same effect.

The circle is the most perfectly symmetrical plane figure, because it can be turned about its centre through any angle whatever or reflected in any diameter. Similarly in three dimensions a sphere can be turned through any angle about any diameter or reflected in any plane through the centre. Apart from figures consisting entirely of concentric circles or spheres there is always a smallest

*Here we exclude the somewhat exceptional case of similarity. See Chapter 10.

possible angle of rotation. The element of pattern can be rotated into 2, 3, 4 or more positions, by half-turns, one-third-turns, quarter-turns, etc. We call these rotations *diad*, *triad*, *tetrad*, and so on, accordingly.

In two dimensions rotation is about a point; in three dimensions about an axis. In either case the isometry is direct. There is another movement, however, called *central inversion* (or simply *inversion*, for short), which in two dimensions is a direct isometry and in three is an opposite one. Inversion is reflexion in a point. Inversion in a point O (called the *centre of inversion*) consists of the replacement of any point A by its image in O, i.e. AO is produced to A' so that $AO = OA'$ (Fig. 1.05). Similarly B is replaced by its image B'.* It is easily seen that in two dimensions inversion is the same as a half-turn, as, for example, in the silhouette pattern of two hands shown in Fig. 1.06(a). But if the hands are three-dimensional, as in Fig. 1.06(b), a right-handed turn of one corresponds to a left-handed turn of the other, showing that inversion in three dimensions is an opposite isometry. It is in fact a combination of rotation and reflexion: a half-turn about an axis and a reflexion in a plane perpendicular to that axis. This can be seen in Fig. 1.07, where A is rotated to A' and then reflected to A''. It can also be demonstrated with two hands.

This shows that it is necessary to consider not only reflexions and rotations but also the various ways in which they can be combined. Moreover, for repeating patterns the basic movements include translations and these will combine with reflexions and rotations to form further movements such as glide reflexions and screw rotations. These we shall consider in due course, dealing first with movements in two dimensions.

Fig. 1.05

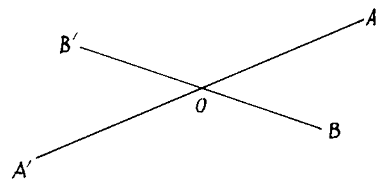


Fig. 1.06

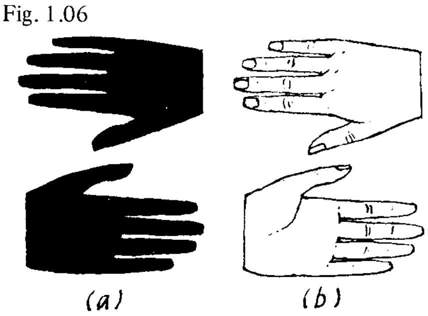
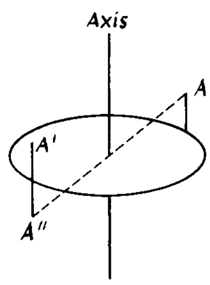
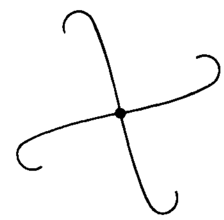


Fig. 1.07



*This transformation is not to be confused with inversion in a circle or a sphere. Algebraically it is $(x, y, z) \to (-x, -y, -z)$, whereas inversion in a sphere is $(x, y, z) \to (k^2/x, k^2/y, k^2/z)$.

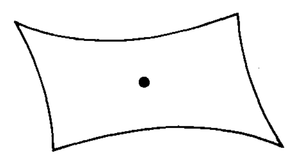
Fig. 2.01



2

Finite patterns in the plane

Fig. 2.02



Looking at finite symmetrical patterns such as those of Figs. 2.01, 2.02 and 2.03, we notice that there is always a centre point or a centre line. In the symmetry movements of these patterns, i.e. in the rotations and reflexions that leave them unchanged, the centre point (or line) remains fixed. The symmetry of such patterns is called *point symmetry*.

By contrast a repeating pattern, as in a frieze or a wallpaper, is built up by translations, which move every point the same distance and in the same direction. The pattern is then in theory infinite.

The only movements that leave a point fixed are rotations about the point and reflexions in lines through it. We consider rotations first and we suppose that there is a minimum angle of rotation. (This excludes designs made up entirely of circles centred on the fixed point.) As a symmetry movement leaves the pattern unchanged it can be repeated any number of times. Thus a turn through 90° means that one of 180° or 270° is equally possible. Hence the minimum angle of rotation must be a sub-multiple of 360°, say $360°/n$, where n is an integer. (For proof of this see p. 106.) Patterns of this sort are said to have *cyclic symmetry*, and there are an infinite number of types, according to the value of n. Figs. 1.04, 2.01, 2.02 show examples in which $n = 3, 4, 2$. These types are sometimes denoted by the symbols $C_3, C_4, C_2, \ldots$, but we shall more often use simply the numbers 3, 4, 2, . . . , according to the agreed 'international' notation for symmetry groups.*

Fig. 2.03

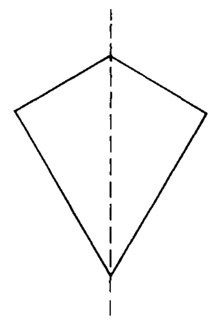
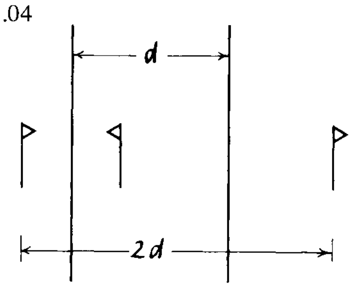


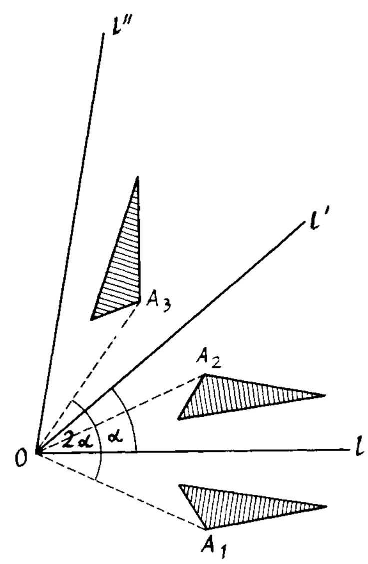
Fig. 2.04



Next we consider reflexions. There may be a single reflexion line, as in Fig. 2.03, and this is perhaps the simplest of all symmetry types. If, however, there is more than one, there must also be a rotation centre. Note first that, for a finite pattern, there cannot be parallel reflexion axes, as that would imply a translation (Fig. 2.04) and hence a repeating pattern. If then the mirror lines intersect at angle α it can be seen from Fig. 2.05 that the two reflexions combine to give a rotation of amount 2α about the point of intersection. (A_1 is reflected in l to A_2 and thence in l' to A_3.)

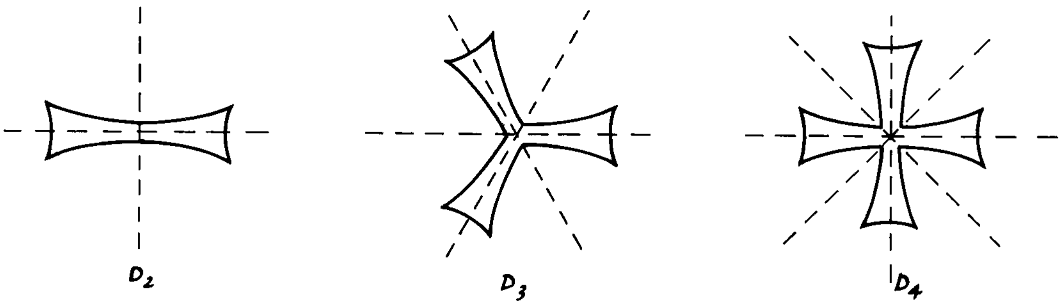
It is equally true, and shown in the same figure, that a rotation about a point and a reflexion in a line through that point combine to give a reflexion in another line through the point. (If A_3 is rotated about O to A_1 and then reflected in l to A_2, the combined movement is a reflexion in l'.) Moreover other reflexion lines are implied, at angular intervals $\alpha, 2\alpha, \ldots$, from l and l'. (If A_3 is rotated as before to A_1 and then reflected in l', the combined movement is reflexion in a line l'' at angle α to l'. Thus the mirror line l is itself reflected to a new mirror line l''.)

Fig. 2.05



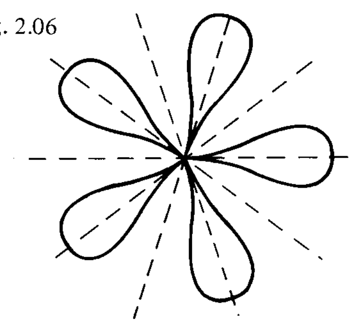
*We shall refer to the symbols C_3, C_4, etc. as being in the *group notation*. See also the Index of Groups.

Fig. 2.07



So, if a pattern admits reflexions in two lines intersecting at O at an angle a, it will also admit rotation about O through $2a$ and reflexions in further lines through O at intervals a. In short, the reflexion lines rotate with the figure, and are reflected in one another (Fig. 2.06).

Fig. 2.06



Patterns of this kind are called *dihedral*. The simplest of these types is that illustrated in Fig. 2.03, with one mirror line. The group symbol for this is D_1, while that of Fig. 2.06, with its pentad rotation and five mirror lines is D_5. Clearly there are an infinite number of dihedral symmetry types. We illustrate D_2, D_3 and D_4 in Fig. 2.07.

The 'international' symbols for these symmetry types give first a number to indicate the rotation, followed by letters m for the mirror lines. It will be noticed that in D_4 the four mirror lines divide into two sets, those along the arms of the cross and those bisecting the angles so formed. Each set is related by the tetrad rotation. This happens with all the even-numbered types, but with D_3, and all odd-numbered types, the two sets coincide. For this reason D_4 is given the symbol $4mm$, but D_3 is $3m$. The two systems may be compared as follows:

D_1	D_2	D_3	D_4	D_5	D_6	. . .
$1m$	$2mm$	$3m$	$4mm$	$5m$	$6mm$	. . .

There are thus two infinite sets of symmetry types for finite patterns in two dimensions, the cyclic types and the dihedral types.