CAMBRIDGE TRACTS IN MATHEMATICS

69. Simple Noetherian rings

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Simple Noetherian rings

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This book is for
Midge and Mickey;
Barbara and Kathy;
Heidi and Cindy

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Preface

This book is intended for the general reader of mathematics, and the authors have eliminated unnecessary mathematical machinery whenever possible, or postponed using it as long as possible. This was done, however, only after the authors saw that it could be done without any loss of clarity, or generality, in the statements of theorems.

The book is as elementary and self-contained as practicable; and the little background required in homological and categorical algebra is listed in two short appendices. Otherwise, full definitions are given, and short, elementary full proofs are supplied for such tool theorems as the Morita theorem 1.12 (p. 15), the correspondence theorem 1.13 (p. 17), the Wedderburn–Artin theorem 2.23 (p. 32), the Goldie–Lesieur–Croisot theorem 4.4 (p. 65), and many others.

The authors are indebted to Professor Hyman Bass, one of the general editors of this series, for suggesting that such a study of simple Noetherian rings would be of interest. We are immensely grateful to Professor Bass and to the Cambridge University Press for making it possible for this study to appear in the Cambridge Tracts in Mathematics.

J.H.C.
C.F.

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List of symbols†

	Name	Example
iff	double implication (or equivalence of propositions)	A is true iff B is true
$\forall$	universal quantifier	$\forall a \in A$
$\exists$	existential quantifier	$\exists a \in A$
$\in$	membership	$a \in A$
$\notin$	nonmembership	$a \notin A$
$\subset$	proper containment	$A \subset B$
$\subseteq$	containment	$A \subseteq B$
$\Rightarrow$	implication	$A \Rightarrow B$
$\Leftrightarrow$	double implication	$A \Leftrightarrow B$
$=$	equals	$A = B$
$\varnothing$	empty set	—
$\mathbb{N}$	natural numbers	$1, 2, \ldots$
$\mathbb{Z}$	integers	$0, \pm 1, \pm 2, \ldots$
$\mathbb{Q}$	rational numbers	a/b, $a, b \in \mathbb{Z}$, $b \neq 0$
$\cup$	union	$A \cup B$
$\cap$	intersection	$A \cap B$
$+$	plus	$a + b$
$-$	minus	$a - b$
$\circ$	composition	$f \circ g$
$\times$	Cartesian product	$A \times B$
$\rightarrow$	mapping	$A \rightarrow B$
$f: A \rightarrow B$	mapping	—
$\mapsto$	mapping	$a \mapsto b$
$\hookrightarrow$	embedding	$A \hookrightarrow B$
$\times$	cross	$\times_i A_i$
$\prod$	product	$\prod_i A_i$
$\coprod$	coproduct	$\coprod_i A_i$

† Much of the material in this table is taken from *Grundlehren* volume 190 and is reproduced by permission of Springer-Verlag, Heidelberg.

$\approx$	equivalence or isomorphism (in the appropriate category)	$A \approx B$, mod-$A \approx$ mod-B
$\simeq$	equivalence (as functors)	$S \simeq T$
$\overset{\text{nat}}{\approx}$	natural isomorphism (or equivalence)	$A \overset{\text{nat}}{\approx} B$
$>$	greater than	$a > b$
$<$	less than	$a < b$
$\geq$	greater than or equal to	$a \geq b$
$\leq$	less than or equal to	$a \leq b$
$\rightsquigarrow$	functor	$A \rightsquigarrow B$
∇	wedge	$A \nabla B$
$\perp$	perpendicular ("perp")	$A^{\perp}$ and ${}^{\perp}A$
$\sum$	summation	$\sum_{i \in I} A_i$
$\oplus$	direct sum	$A \oplus B$, $\sum_{i \in I} \oplus A_i$
$\otimes$	tensor product	$A \otimes B$
$\square$	end of proof (occasionally, no proof)	—
$\sim$	similarity (Morita equivalence)	$A \sim B$
$\overset{Q}{\sim}$	equivalence (as orders)	$A \overset{Q}{\sim} B$
$\overset{l}{\sim}$	left equivalence (as orders)	$A \overset{l}{\sim} B$
$\overset{r}{\sim}$	right equivalence (as orders)	$A \overset{r}{\sim} B$

mod-A = the category of (unital) right A-modules
A-mod = the category of left A-modules
Ab = the category of abelian groups (mod-$\mathbb{Z}$)
L.H.S. (R.H.S.) = left (right) hand side
$\{x \mid x \text{ has property } P\}$ = the set of all x with property P
$\{x_i \mid i \in I\}$ = a set of elements indexed by a set I

Conventions

The following conventions are the standard ones adopted by most modern texts on homological algebra or ring theory. Their inclusion is primarily for the reader's convenience.

For the most part, by a ring A, we mean an associative ring with unit, usually denoted 1. We let ring-1 denote a ring in which an identity element is not assumed. Thus, a ring-1 may have 1. All ring homomorphisms will be assumed to preserve the respective units. We let mod-A (A-mod) denote the category of all unital right A-modules (left A-modules), the morphisms being right (left) A-linear homomorphisms. The fact that M is a right (left) A-module will be denoted variously by $M \in$ mod-A ($M \in A$-mod) and M_A (${}_AM$).

$f: M_A \to N_A$ ($f: {}_AM \to {}_AN$) will indicate that f is a morphism (A-linear homomorphism) between right A-modules (left A-modules) M and N.

If $f: M_A \to N_A$ ($f: {}_AM \to {}_AN$), we shall write f on the left (right) of its argument. Thus $f(m)$ ($(m)f$) will denote the image of f on $m \in M_A$ ($m \in {}_AM$).

If $M, N \in$ mod-A and $\hom_A(M_A, N_A) = \{f \mid f: M_A \to N_A\}$, we set $\text{End } M_A = \hom_A(M_A, M_A)$. Similarly, $\text{End } {}_AM = \hom_A({}_AM, {}_AM)$. Of course, End M_A (End ${}_AM$) is called the **endomorphism ring** of M_A (${}_AM$).

Whenever $M \in$ mod-A and $M \in B$-mod and $b(ma) = (bm)a$, $\forall m \in M, a \in A, b \in B$, we call M a (B, A)-bimodule and denote this variously by $M \in B$-mod-A or ${}_BM_A$. The morphisms in the category B-mod-A are homomorphisms $f: M \to N$ which satisfy

$$f(bma) = bf(m)a \quad \forall m \in M, a \in A, b \in B.$$

Let $M \in$ mod-A and $B = \text{End } M_A$. Since we have agreed to write homomorphisms on the left of their arguments, M is naturally a left B-module, in fact a (B, A)-bimodule where

$b \cdot m = b(m)$, $\forall b \in B$, $m \in M$. Similarly, if $M \in A$-mod and $B = \text{End}\,_A M$, $M \in A$-mod-B.

If $M \in B$-mod and $A = \text{End}\,_B M$ then $\bar{B} = \text{End}\, M_A$ is called the **biendomorphism ring (bicommutator)** of M and is frequently denoted by $\bar{B} = \text{Biend}\,_B M$. Clearly, B maps homomorphically into $\bar{B}$ (as rings) via left multiplication. This map is obviously injective whenever $_BM$ is faithful. If the map is a surjection $_BM$ is called **balanced**. In this case, center $A \approx$ center B via the canonical map of B into Biend $_BM$.

A **basis** is another term for a generating set of a module. However, **free basis** means that a basis $\{x_i \mid i \in I\}$ is **linearly independent** in the sense that any (finite) linear combination $\sum_{i \in I} x_i r_i$ is zero iff $r_i = 0$ for every i. (Finite here means that $r_i = 0$ for all but a finite number of $i \in I$.) In this case, the sum $\sum_{i \in I} x_i R$ of the submodules generated by the set $\{x_i \mid i \in I\}$ is direct, notationally, $\sum_{i \in I} \oplus\, x_i R$.

As indicated in the list of symbols, $M = A \oplus B$ denotes a **direct sum** of modules A and B, and then A (also B) is said to be a **direct summand** of M, or more briefly, a **summand** of M. (Thus, every $m \in M$ can be expressed in one and only one way, $m = a + b$, as a sum of elements $a \in A$ and $b \in B$.)

In order to save space, in defining 'right-handed' concepts, we shall assume the left-handed concept defined by right-left symmetry. (Similarly for left-handed concepts.) Example: right Noetherian.

A mapping $f: X \to Y$ of sets X and Y is said to be **injective**, or an **injection** provided that $f(x) = f(y) \Rightarrow x = y$, $\forall x, y \in X$. It can be shown that a mapping $f: X \to Y$ is injective iff f is **left cancelable** in the category of set morphisms in the sense that for any two mappings $g: U \to X$ and $k: U \to X$, then $fg = fk$ iff $g = k$.

Dually, a mapping $f: X \to Y$ is **surjective**, or a **surjection**, iff $Y = \text{im}\, f$. This happens iff f is **right cancelable**.

The terms map and mapping are synonyms for morphism in the specified category. Thus, a homomorphism of groups is also called a map. We use **monomorphism**, or the abbreviation, **mono**, in the sense that the morphism is left cancelable in the category; dually for **epimorphism**, or **epi**. Thus, an injection is mono in the category SETS of sets, a surjection is epi, and conversely. Similarly, in Ab (the abbreviation for mod-$\mathbb{Z}$, the category of abelian groups), a morphism $f: X \to Y$ is mono iff $\ker f = 0$ iff f is injective, and dually, f is epi iff $\operatorname{cok} f = 0$ iff f is surjective. Here, of course, $\ker f$ denotes the kernel of f, and $\operatorname{cok} f$ denotes the cokernel (which is isomorphic to $Y/\operatorname{im} f$).

The term 'dually' referred to above is meant in the sense that one concept is obtained from the other by reversing arrows and directions in the definitions. For a more explicit statement of duality, consult Faith [73*a*, p. 60].

A sequence $X \to Y \to Z$ of modules is **exact** if the image of $X \to Y$ is the kernel of $Y \to Z$. This terminology also applies to long sequences of modules and maps, namely

$$\ldots \to X_i \to X_{i+1} \to X_{i+2} \to \ldots \to X_n \to X_{n+1} \to \ldots$$

is **exact**, provided that each $X_i \to X_{i+1} \to X_{i+2}$ is exact for every i, with possibly i ranging over $\mathbb{Z}$. An exact sequence

$$0 \to X \to Y \to Z \to 0$$

is said to be **short exact**. Thus, $0 \to X \to Y$ is exact iff $X \to Y$ is mono, and $X \to Y \to 0$ is exact iff $X \to Y$ is epi. Thus there is always a canonical exact sequence

$$0 \to X \to Y \to Y/X \to 0$$

for any submodule X of Y.

Other requirements from homological algebra will be administered *ad hoc*, or, in a few cases, found in the Appendix, §1 and §2. The term **natural isomorphism** is an old one which has been subsumed under the concept of natural equivalence of functors defined in the Appendix, §2, but often we use it in its old informal sense to mean that the mapping indicated by $\overset{\text{nat}}{\approx}$ is the canonical one, and is an isomorphism.

A module M of mod-A is **Noetherian** if it satisfies the equivalent conditions:

(N_1) **a.c.c.**: Any ascending sequence

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$$

of submodules is ultimately stationary, that is, there exists an integer $N > 0$ such that $M_N = M_{N+k}\ \forall k \geq 1$.

(N_2) M has the maximum condition for submodules, that is, any nonempty set S of submodules contains a maximal element $M(S)$. (Thus, $M(S) \not\subset M'$ for any $M' \in S$.)

(N_3) Every submodule of M has a finite basis, that is, is finitely generated.

Dual to the concept of Noetherian is the concept of **Artinian**, namely, a right A-module M is Artinian provided that M satisfies the equivalent conditions:

(A_1) **d.c.c.** Any descending sequence

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

of submodules is ultimately stationary.

(A_2) M satisfies the minimum condition for submodules.

The ring itself is **right Noetherian (Artinian)** provided that A is a Noetherian (Artinian) module in mod-A. This happens iff every finitely generated module in mod-A is Noetherian (Artinian), a fact which shows that this is a Morita invariant property (in the sense defined on p. 15.) Furthermore, a theorem of Hopkins and Levitzki states that any right Artinian ring A is right Noetherian. (Since this result for simple rings is an immediate consequence of the Wedderburn–Artin theorem 2.23, p. 32, we omit the proof.)

The **Hilbert Basis Theorem** states that if A is a right Noetherian ring, then so is the polynomial ring $A[X]$ over A. This readily generalizes to finitely many variables. (See, for example, Faith [73*a*], p. 341.)

A module N is **simple** if N has no nontrivial submodules. For $M \in$ mod-A, the **socle** of M, soc M, is the sum of all simple submodules of M. If soc $M = M$, we call M **semisimple**. 'Dually', a submodule N of M is a **maximal submodule** of M or simply maximal, if M/N is simple, and the **radical** of M, rad M, is the intersection of all maximal submodules of M.

An ideal P of A is called **prime (semiprime)** if for all ideals I and J of A, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$ ($I^2 \subseteq P \Rightarrow I \subseteq P$). Here and hereafter, by an ideal of A, we mean a two-sided ideal of the ring A. A ring A is **prime** (**semiprime**) if 0 is a prime (semiprime) ideal of A.

A ring A is **simple** provided that there are no ideals except the trivial ones, A and 0. Clearly, every simple ring is prime (0 is a maximal ideal!). Simplicity of a ring A may be characterized by the property that every right module $M \neq 0$ is faithful, a Morita invariant property which shows that any full $n \times n$ matrix ring A_n over a simple ring A is simple.

The term **domain** is an abbreviation for an integral domain, that is, a ring without zero divisors $\neq 0$. Finally, a **field**, defined as a ring in which every element $x \neq 0$ has a (two-sided) inverse x^{-1}, may be characterized by the property that the only right ideals are the trivial ones. Thus, a field is simple.