Introduction

In this book we have collected the known theorems on the structure of simple right Noetherian rings, and more generally for simple rings containing a uniform right ideal U. By the Goldie [58, 60]† and Lesieur–Croisot [59] theorems, any prime right Noetherian ring R has a simple Artinian ring $Q(R)$ of right quotients. By the Artin [27]–Wedderburn [08] theorem, $Q(R)$ is isomorphic to a full $n \times n$ matrix ring D_n over a (not necessarily commutative) field D. Moreover, the endomorphism ring B of any uniform right ideal U of R is a right Ore [31] domain with right quotient field $Q(B) \approx D$.

The Goldie and Lesieur–Croisot theorem is historically the first representation of a right Noetherian non-Artinian prime ring in a matrix ring of finite rank with entries in a field. (The Chevalley–Jacobson density theorem for a primitive ring R yields a representation by infinite matrices over a field unless R is Artinian.)

Significant simplifications in, and additions to, the Goldie and Lesieur–Croisot theory occur if R is assumed to be a simple ring. Henceforth R will denote a simple ring with identity element, and uniform right ideal U as described. We now outline the structure theory for R, which includes that for a simple right Noetherian ring.

1 *Endomorphism ring theorem* 2.17‡

Every simple ring R with uniform right ideal U is the endomorphism ring, $R \approx \text{End}_B U$ canonically, of a torsion free

† Goldie [58] refers to a work by A. W. Goldie published in 1958; co-authors are referred to in the text in hyphenated form, in conformity with the widespread practice in mathematics.

‡ The notation 2.17 refers to a theorem in Chapter 2 numbered 17. In general, for integers a and b, the notation a.b will denote a statement numbered b in Chapter a.

module ($=U$) of finite rank over a right Ore domain B (Faith [64]).

2 *Projective module theorem* 2.17

U is a finitely generated projective left B-module, where $B = \text{End } U_R$ (Hart [67]).

3 *Uniqueness of orders theorem* 2.22

Assume that $R \approx \text{End }_{B'}U'$ for a finitely generated projective module U' over a right Ore domain B'. Then there is an embedding $B' \to D$ such that B' and B are equivalent orders of D. (Equivalence of orders is defined in the sense of Jacobson [43] in Chapter 2 (p. 26).)

In the next theorem, $\text{trace }_{B'}U' = \sum_{f \in \text{Hom}_{B'}(U', B')} (U')f$, that is, the B'-submodule generated by all elements $(u)f$, with $u \in U'$, $f \in \text{Hom}_{B'}(U', B')$.

4 *Least ideal theorem* 2.6

In any representation $R \approx \text{End }_{B'}U'$ (as in 3), $T' = \text{trace }_{B'}U'$ is a (nonzero) least ideal of B'.

Thus, $T' = B'$ iff B' is a simple ring.

5 *Idempotent right ideal theorem* 2.21

Moreover, R is right Noetherian iff B' (as in 4) satisfies the ascending chain condition (a.c.c.) on idempotent right ideals contained in T'. In this case B satisfies the a.c.c. on principal right ideals.

6 *Single ideal theorem* 2.7

There is a module U' and domain B' (as in 3) such that $R \approx \text{End }_{B'}U'$ and such that B' contains at most one nontrivial ideal T'.

7 *Same centers theorem* 2.7

The center C of B is a field canonically isomorphic to the center of R, and U' and B' (in 6) can be chosen such that $U' = U$ and $B' = C + T$, where $T = \text{trace}\,_B U$.

In other words, U is finitely generated projective over $B' = C + T$, and $R \approx \text{End}\,_{B'}U$ canonically. Moreover, center $B' = C$ and trace $_{B'}U = T$.

8 *Hereditary ring theorem* 2.25

Every representation of R as an endomorphism ring $R \approx \text{End}\,_{B'}U'$ with $B' = \text{End}\,U'_R$ (as in 3) has the property that B' is a simple ring iff R is right hereditary.

This shows that the single ideal theorem is of some use: the representation of R over a domain with a single nontrivial ideal does occur (as in 6) when R is a nonhereditary simple ring.

The results 3 to 8 are proved by Faith [72*a*] and are consequences of the correspondence theorem for projective modules taken up in Chapter 1.

Any simple Noetherian ring R is a maximal order in $Q(R)$, in a sense defined in Chapter 2 (p. 27), and the question arises when is $B = \text{End}\,U_R$ a maximal order in D. When R is a two-sided order we have:

9 *Maximal order theorem* 4.14

The domain $B = \text{End}\,U_R$ is a maximal order in D iff U is a reflexive right R-module. This happens, for example, when U is a maximal uniform (=basic) right ideal.

By a theorem of Goldie, any uniform right ideal Y of R is contained in a basic right ideal. This, together with the previous theorems shows that R can be represented as the endomorphism ring of a finitely generated projective over a maximal order in a field.

10 *Reflexive ideal theorem* 4.20

If R is two-sided Noetherian with right global dimension $R \leq 2$, then every uniform right ideal of R is reflexive iff R is right hereditary.

Theorems 9 and 10 are proved by Cozzens [75], and taken together, Theorems 1 to 10 reduce many structural questions on a simple right Noetherian ring R to corresponding questions on the structure of a maximal right order B (of a field D) having at most one nontrivial ideal. For example, the question of when U can be chosen such that $B = \text{End } U_R$ is itself a simple ring can be formulated entirely within D, namely

11 *Simple endomorphism rings theorem* 2.20

Assume that R has the representation $R = \text{End }_{B'}U'$ as in 3. Then, there exists a simple right Ore domain B_0 and a finitely generated projective left B_0-module U_0 such that $R \approx \text{End }_{B_0}U_0$ iff there exists a right ideal I of B' such that

$$(I : I) = \{d \in D \mid dI \subseteq I\}$$

is a simple subring of D. (Note that since End $I_{B'} \approx (I : I)$, this is equivalent to requiring that some right ideal of B' has simple endomorphism ring.) (Faith [72*a*].)

The question of the existence of such a representation of R (as in 11) is one of the main structural problems on a simple right Noetherian ring R, and in this case (and only then) R is similar (= Morita equivalent) to a right Ore domain A, that is, then (and only then) there is a category equivalence mod-$R \approx$ mod-A, for some domain A. A positive result in this direction is the following:

12 *Global dimension* 2 *theorem* 2.40

A simple Noetherian ring R of right global dimension ≤ 2 is similar to a (simple) domain A. (Faith [72*a*], Hart–Robson [70], Michler [69]). The proof requires ideas behind Bass's global dimension 2 theorem (4.16).

This holds if R is a principal right ideal ring, since then r.gl.dim $R \leq 1$, but in this case much more is true:

13 *Principal right ideal theorem* 4.7

Any prime principal right ideal ring R is isomorphic to a full $n \times n$ matrix ring A_n over a right Noetherian hereditary domain A. (Goldie [62].)

Curiously, A need not be a principal right ideal domain for this to happen (Swan [62], see also 3.6, p. 49). (When R is simple then of course A is.)

14 *Ore domains and the Faith–Utumi theorem* 4.6

A refinement of the Goldie–Lesieur–Croisot theorem is the theorem of Faith–Utumi (proved in Chapter 4) which states that if R is any right order in D_n, then R contains a right order F_n, where F is a right order in D. A change of matrix units in D_n may be necessary for this (unless R is also a left order) and, moreover, F does not in general contain an identity element. [If it did, then R would contain the full set of matrix units of D_n, and hence R would itself be a full $n \times n$ matrix ring H_n, where $H = R \cap D$. This is, in general, not the case.] One can show that F can be chosen to contain no nontrivial ideals when R is simple.

15 *Classification of simple domains*

Inasmuch as Ore domains play a role in the structure of simple Noetherian rings analogous to the role of fields in the structure of simple Artinian rings, their classification, especially the simple ones, is fundamental to the structure theory of simple Noetherian rings. The rest of the introduction is devoted to this.

Restricting our attention for the moment to simple integral domains, we can distinguish three main types:

(D_1) For any simple domain (or ring) R of characteristic 0, and finitely many commuting outer derivations $\delta_1, \ldots, \delta_n$, the ring $\mathscr{D}_R$ of differential polynomials is a simple right Noetherian Ore domain. (Ore [33], Littlewood [33], Amitsur [57].) We call these **Amitsur–Littlewood** (simple Ore) **domains.**

(D_2) For any field F, the localization R_M at a maximal ideal M, of the skew polynomial ring $R = F[x]$ with respect to a ρ-derivation δ is a simple principal left ideal (pli-) domain, not a field. (Jacobson [43].) We call these domains **simple localizations.**

(D_3) For any field F, if $F[x]$ is a primitive ring, e.g., if F is a transcendental algebra over the center C, then $A = F \otimes {}_C C(x)$ is a simple pli and pri (principal right ideal)-domain.

D_3 is an observation derivable from Jacobson's theorem [64*a*] characterizing when $F[x]$ is a primitive ring. (See Theorem 3.20 and 3.21, pp. 61–2.) For this reason, we entitle this class **Jacobson domains.** (Actually, D_3 is a subclass of D_2.)

16 *V-domains*

In this the concluding section of the Introduction, we summarize some results on a special class of simple Ore domains for which there is a rather satisfactory structure theory paradoxically resembling (commutative) Dedekind domains.

The question of the existence of integral domains, not fields, having the property that every simple right module is injective was first raised by Faith [67*a*, Problem 17, p. 130]. The history of this problem is as follows: Kaplansky showed that commutative regular rings could be characterized by the injectivity of the simple modules. We use the terminology right K-ring (after Kaplansky) for a ring with every simple right module injective. Villamayor characterized a right K-ring R by the property that every right ideal is the intersection of maximal right ideals, or equivalently, every right module M has the property that the intersection of its maximal submodules, namely rad M, is equal to 0. Such rings are called right V-rings, and we refer interchangeably to right K-rings and V-rings.

The first examples of right V-domains, not fields, given by Cozzens [70], were differential polynomial rings $K[x, d]$ with respect to a derivation d over a (Kolchin) universal differential field K. Analogous examples of these right V-domains were localizations of twisted polynomial rings $K[x, \sigma]$ with respect to an automorphism σ of a field K over which certain designated algebraic equations have solutions. These are taken up in Chapter 5.

More generally, simple Noetherian V-domains, not fields, were shown to exist in all homological dimensions by Cozzens–Johnson [72]. Osofsky has shown the existence of V-domains having infinitely many nonisomorphic simple modules (see 6.20). However, the problem of exhibiting examples with just finitely many nonisomorphic simple modules remains open.

All of the above examples which are pri-domains also have the property that each cyclic module, not isomorphic to the base ring, is injective. Such rings are called PCI-rings (proper cyclics injective). Chapter 6 is devoted to the study of these rings: they are either semisimple Artinian, or else semihereditary simple right Ore domains in which any finitely generated right ideal I can be generated by two elements, one of which is an arbitrary nonzero element of I. Other resemblances to Dedekind domains are noted.

1. *The correspondence theorem for projective modules*

In this chapter, we prove the correspondence theorem for projective modules, referred to in the Introduction, on which most of the known structure theory for simple Noetherian rings depends.

If $M, N \in$ mod-A and $B = \text{End } M_A$, then **the trace of** N **in** M is the A-submodule of M generated by all elements $\{f(x) \mid x \in N, f \in \text{hom}_A(N, M)\}$. This module is denoted trace ${}_M N$ and is a (B, A)-bimodule. In particular, trace ${}_A M$ is an ideal of A for any module M.

1.1 Proposition and Definition *An object* U *of* mod-A *is a* **generator** *if the equivalent conditions hold:*

(1) *The group-valued functor* $\text{hom}_A(U, -) : \text{mod-}A \rightsquigarrow \text{Ab}$ (*the category of abelian groups*) *is faithful. Thus, given any nonzero* $k: X_A \to Y_A$ *there exists a homomorphism* $h: U_A \to X_A$ *such that* $kh \neq 0$.

(2) trace ${}_M U = M$ *for all* $M \in$ mod-A.

(3) trace ${}_A U = A$.

(4) $U^n \approx A \oplus X$ *for some integer* $n > 0$ *and* $X \in$ mod-A.

(5) $U^n \to A \to 0$ *is exact for some* $n > 0$.

Proof For any index set I, $U^{(I)}$ will always denote the direct sum (coproduct) of I copies of U.

(1) ⇒ (2): Define

$$\pi \begin{cases} U^{(\text{hom}_A(U,M))} \to M, \\ (\ldots, u_f, \ldots) \mapsto f(u) \quad \forall f \in \text{hom}_A(U, M). \end{cases}$$

Note that im π = trace ${}_M U$. If im $\pi \neq M$, set $C = M/\text{im } \pi$ and let $\nu: M \to C$ be the canonical projection. By assumption, there exists an $f \in \text{hom}_A(U, M)$ such that $\nu f \neq 0$. However, im $f \subseteq$ im π, contradicting $\nu f \neq 0$. Thus, im $\pi = M$.

(2) ⇒ (1): Let $g: M \to N$ be nonzero. If $gf = 0$ for all $f \in \text{hom}_A(U, M)$, then clearly $g\pi = 0$ implying that $g = 0$ by surjectivity of π. Thus, $gf \neq 0$ for some $f \in \text{hom}_A(U, M)$.

(2) ⇒ (3) is clear; the equivalence of (3), (4) and (5) is immediate by 1.2 which follows.

(3) ⇒ (1): Since A is free, there exists an epi $A^{(I)} \to M$ for some set I and hence, an epi $U^{(I)} \to M$. By the same arguments used in the proof (2) ⇒ (1), (1) is clearly satisfied. □

1.2 Proposition and Definition *An object* $U \in$ B-mod *is* **projective** *if the following equivalent conditions hold:*

(1) *The group valued functor* $\hom_B(U, -) : B\text{-mod} \rightsquigarrow \text{Ab}$ *is exact.*

(2) (Dual basis lemma) *There exist sets* $\{x_i \mid i \in I\}$ *of elements of* U *and* $\{f_i \mid i \in I\}$ *of elements of* $\hom_B(U, B)$ *such that for each* $x \in U$, $(x)f_i = 0$ *for almost all* $i \in I$ *and*

$$x = \sum_{i \in I} (x)f_i x_i \quad \forall x \in U.$$

(3) *U is isomorphic to a (direct) summand of a free B-module.*

(4) *Every exact sequence* $M \to U \to 0$ *in* B-mod *splits.*

Proof The equivalence of (1), (3) and (4) is elementary, and will be left for the reader. We shall contend with the equivalence of (2) and (3).

(4) ⇒ (2): Let $\{x_i \mid i \in I\}$ be *any* generating set for ${}_BU$, $\{e_i \mid i \in I\}$ a basis for $B^{(I)}$ and $f : B^{(I)} \to U$ the B-map defined by $e_i \mapsto x_i$. Since f splits there exists a map $g : U \to B^{(I)}$ such that $gf = 1_U$.

For any $x \in U$,

$$(x)g = \sum_{i \in I} b_i e_i,$$

where $b_i = 0$ for almost every $i \in I$. Define a family $\{f_i \mid i \in I\}$, where each $f_i : {}_BU \to {}_BB$, by $x \mapsto b_i$ for all $x \in U$. Clearly,

$$\begin{aligned} x &= \sum b_i x_i \\ &= \sum (x)f_i x_i \qquad \forall x \in U. \end{aligned}$$

(2) ⇒ (3): Let $f : B^{(I)} \to U$ be defined as before and define $g : U \to B^{(I)}$ by

$$(u)g = \sum_{i \in I} (u)f_i e_i \quad \forall u \in U.$$

Clearly, g is a B-map and $gf = 1_U$. □

Remarks (*a*) The proof shows card I can be chosen equal to the cardinal of any basis of U, and, moreover, the equation for x in (2) shows that $\{x_i \mid i \in I\}$ is a basis of U.

(*b*) The set $\{f_i \mid i \in I\}$ is a basis of the 'dual' module $\mathrm{Hom}_B(U, B)$ if ${}_BU$ is finitely generated.

(*c*) If A is any simple ring, then any right ideal $U \neq 0$ is a generator of mod-A, for $T = \text{trace}\ {}_AU$ is an ideal containing AU, hence $T = A$.

1.3 Proposition *If U is projective in B-mod, then*

$$T = \text{trace}\ {}_BU = \bigcap_{\substack{K \subseteq B \\ \text{and } KU = U}} K.$$

That is, T is the intersection of all right ideals K of B such that $KU = U$. In particular, $U = TU$. Moreover, $T^2 = T$.

Proof By (2) of 1.2, $U = TU$ since each $(u)f_i \in T$. Thus, the L.H.S. $\supseteq$ R.H.S. Conversely, if $U = IU$, for some (right) ideal of B, then

$$T = \text{trace}\ U = \text{trace}(IU) = IT \subseteq I.$$

Thus, R.H.S. $\supseteq$ L.H.S. The fact that $T^2 = T$ follows immediately from the dual basis lemma. □

1.4 Cut-Down Proposition *Let U be projective*

$$B^{(I)} \xrightarrow{p} U \to 0$$

exact in B-mod, and let $T = \text{trace}\ {}_BU$. If B_0 is any subring of B containing T, then U is projective in B_0-mod and p induces an exact sequence

$${B_0}^{(I)} \to U \to 0.$$

Furthermore, the inclusion map $\text{End}\ {}_BU \to \text{End}\ {}_{B_0}U$ is an isomorphism. If U is faithful over B, then

$$\text{center}\ B_0 = B_0 \cap \text{center}\ B.$$

Proof Let $T^{(I)} \subseteq {B_0}^{(I)}$ be the canonical inclusion. Then,

$$p({B_0}^{(I)}) \supseteq p(T^{(I)}) = Tp(B^{(I)}) = TU = U$$

that is, $p \mid {B_0}^{(I)}$ is surjective. Since ${}_BU$ is projective, there exists a map $s : U \to B^{(I)}$ satisfying $sf = 1_U$. Clearly, im $s \subseteq$