CHAPTER I

THE POINT, THE STRAIGHT LINE AND THE PLANE

1. Knowledge assumed. This book is an immediate sequel to the author's earlier volume *The Methods of Plane Projective Geometry based on the use of General Homogeneous Coordinates* (Cambridge University Press, 1946), and knowledge of the subject-matter will be assumed. We also quote certain obvious extensions to space of three dimensions without giving detailed proofs.

As in the book just named, we again confine our first attention to projective geometry, in which the ideas of length and angle are not used. There are now four homogeneous coordinates instead of three, not necessarily real. The basis on which the use of complex coordinates rests is closely analogous to the development given for plane geometry, but for a rigorous treatment the reader should consult a more advanced text-book.

2. The homogeneous coordinates. We make the following basic assumption:

The position of a point can be uniquely defined by the ratios of four coordinates x, y, z, t, and, conversely, these ratios define a point uniquely.

We assume that x, y, z, t are not all zero simultaneously. Apart from that, they may take any sets of values, real or complex. In speaking of the point with coordinates x, y, z, t we shall often call it simply 'the point (x, y, z, t)'; if we have given the point a name, say P, we shall speak of it as '$P(x, y, z, t)$'.

3. The symbol of a point. It is sometimes convenient to have a single composite sign for the coordinates (x, y, z, t) of a point P, and we use the notation $\mathbf{P}$ for this purpose.* We call $\mathbf{P}$ the *symbol* of the point $P(x, y, z, t)$, and sometimes speak of '$\mathbf{P}$' as a point, meaning 'the point P whose symbol is $\mathbf{P}$'.

* The reader who is familiar with matrices may read $\mathbf{P}$ as denoting the column matrix $\{x, y, z, t\}$. This idea will be developed in Chapter VIII.

If $P_1, P_2, \ldots, P_n$ are n given points, and if non-zero numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be found such that

$$\lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \ldots + \lambda_n \mathbf{P}_n \equiv \mathbf{0},$$

by which we mean that there exist relations

$$\begin{aligned} \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n &= 0, \\ \lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n &= 0, \\ \lambda_1 z_1 + \lambda_2 z_2 + \ldots + \lambda_n z_n &= 0, \\ \lambda_1 t_1 + \lambda_2 t_2 + \ldots + \lambda_n t_n &= 0 \end{aligned}$$

connecting the coordinates, then we say that the points

$$P_1, P_2, \ldots, P_n$$

are *in syzygy*; the identity itself is called a *syzygy*.

If $n = 2$, the syzygy between P_1 and P_2 implies that they are in fact the same point. We shall always assume that no two points are in syzygy unless the contrary possibility is stated explicitly.

4. The straight line. Let $A(x_1, y_1, z_1, t_1)$, $B(x_2, y_2, z_2, t_2)$ be two given points. *We define the line AB to consist of the points*

$$P(x, y, z, t)$$

for which values of the ratio $\lambda : \mu$ *can be found such that*

$$\begin{aligned} x &= \lambda x_1 + \mu x_2, \\ y &= \lambda y_1 + \mu y_2, \\ z &= \lambda z_1 + \mu z_2, \\ t &= \lambda t_1 + \mu t_2. \end{aligned}$$

Every value of $\lambda : \mu$ determines one and only one point,* which is called *a point of the line AB*. In particular, $\mu = 0$ determines A and $\lambda = 0$ determines B. The two values λ, μ cannot vanish simultaneously.

* It is, of course, understood that the coordinates (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2), and not merely the ratios, are completely settled before applying the definition.

The four equations just given can be expressed compactly in terms of the symbols, in the form

$$\mathbf{P} \equiv \lambda\mathbf{A} + \mu\mathbf{B}.$$

The identity between the symbols of three points on a straight line is thus a syzygy. More symmetrically, if the symbols **A**, **B**, **C** of three points A, B, C are connected by means of the syzygy

$$\lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C} \equiv \mathbf{0},$$

then each point lies on the line joining the other two, and the three points are collinear.

Theorem-examples.* **1.** A straight line is determined by *any* two of its points. [Compare M., p. 5.]

2. If the symbols of four points A, B, C, D are connected by means of a syzygy

$$\lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C} + \rho\mathbf{D} \equiv 0,$$

then the lines AD, BC intersect (or, in a special case, coincide).

[The symbol of their common point is a multiple of either $\lambda\mathbf{A} + \rho\mathbf{D}$ or $\mu\mathbf{B} + \nu\mathbf{C}$.]

3. The two points whose symbols are $\lambda\mathbf{A} \pm \mu\mathbf{B}$ separate A, B harmonically.

[The parameter λ/μ in the symbol $\lambda\mathbf{A} + \mu\mathbf{B}$ determines the individual points of the line, and the cross-ratio of four points is the cross-ratio of the four corresponding parameters.]

4. If O, A, B, C are four given points and A', B', C' three given points on OA, OB, OC respectively, then, by adjusting multiples of **A**, **B**, **C** when necessary, the symbols of A', B', C' can be taken, without loss of generality, in the respective forms

$$\mathbf{O} + \mathbf{A}, \quad \mathbf{O} + \mathbf{B}, \quad \mathbf{O} + \mathbf{C}.$$

5. The plane. Let

$$A(x_1, y_1, z_1, t_1), \quad B(x_2, y_2, z_2, t_2), \quad C(x_3, y_3, z_3, t_3)$$

be three given NON-COLLINEAR points. *We define the plane ABC to consist of the points $P(x, y, z, t)$ for which values of the ratios $\lambda:\mu:\nu$ can be found such that*

$$x = \lambda x_1 + \mu x_2 + \nu x_3,$$
$$y = \lambda y_1 + \mu y_2 + \nu y_3,$$
$$z = \lambda z_1 + \mu z_2 + \nu z_3,$$
$$t = \lambda t_1 + \mu t_2 + \nu t_3.$$

* The Theorem-examples are a basic part of the text, and should be both solved and remembered.

Every set of values of $\lambda:\mu:\nu$ (not all zero) determines one and only one point, which is called *a point of the plane ABC*. In particular, $\mu = \nu = 0$ determines A, $\nu = \lambda = 0$ determines B and $\lambda = \mu = 0$ determines C. The three values λ, μ, ν cannot vanish simultaneously.

The four equations just given can be expressed compactly in terms of the symbols, by means of the syzygy

$$\mathbf{P} \equiv \lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C}.$$

The coordinates of P satisfy the relation, found by eliminating $-1:\lambda:\mu:\nu$,

$$\begin{vmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ t & t_1 & t_2 & t_3 \end{vmatrix} = 0.$$

This relation, on expansion, assumes the form

$$lx + my + nz + pt = 0,$$

where l, m, n, p are the cofactors of x, y, z, t in the determinant. This is a homogeneous linear equation in the four variables x, y, z, t, and is called the *equation of the plane*.

Conversely, every point $P(x, y, z, t)$ whose coordinates satisfy the relation

$$\begin{vmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ t & t_1 & t_2 & t_3 \end{vmatrix} = 0$$

does lie in the plane ABC. For there then exist numbers p, q, r, s such that

$$px + qx_1 + rx_2 + sx_3 = 0, \text{ etc.};$$

moreover, p cannot be zero, otherwise the points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ would be connected by a syzygy $q\mathbf{A} + r\mathbf{B} + s\mathbf{C} \equiv \mathbf{0}$ and so be collinear, contrary to the hypothesis. Dividing by p and writing $q/p = -\lambda$, $r/p = -\mu$, $s/p = -\nu$, we find the relation expressed concisely in the form

$$\mathbf{P} \equiv \lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C},$$

which shows, by definition, that P lies in the plane ABC.

It is customary to interchange rows and columns and to write the *equation of the plane ABC* in the form

$$\begin{vmatrix} x & y & z & t \\ x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_3 & y_3 & z_3 & t_3 \end{vmatrix} = 0.$$

An immediate corollary of the preceding work is that, *if the points* (x_i, y_i, z_i, t_i), *with* $i = 1, 2, 3, 4$, *are coplanar, then*

$$\begin{vmatrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_3 & y_3 & z_3 & t_3 \\ x_4 & y_4 & z_4 & t_4 \end{vmatrix} = 0.$$

Conversely, *if this determinant vanishes, then the points are coplanar.*

Theorem-examples. 1. If the symbols of four distinct points A, B, C, D are connected by the syzygy

$$\lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C} + \rho\mathbf{D} \equiv 0,$$

then each point lies in the plane determined by the other three, and the four points are coplanar.

How is this result modified if **A**, **B**, **C** are themselves connected by a syzygy?

2. A plane is determined by *any* three of its points which are not collinear. [Use an argument similar to M., p. 5.]

3. Every linear equation

$$lx + my + nz + pt = 0,$$

where l, m, n, p are not all zero, does determine a plane, and determines it uniquely.

[Three points of the plane are $(-p, 0, 0, l)$, $(0, -p, 0, m)$, $(0, 0, -p, n)$. Compare M., p. 6.]

4. The equations

$$l_1x + m_1y + n_1z + p_1t = 0, \quad l_2x + m_2y + n_2z + p_2t = 0$$

determine the same plane if $l_1 : l_2 = m_1 : m_2 = n_1 : n_2 = p_1 : p_2$.

5. A *necessary and sufficient* condition that the four planes

$$l_i x + m_i y + n_i z + p_i t = 0 \quad (i = 1, 2, 3, 4)$$

have a common point is

$$\begin{vmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{vmatrix} = 0.$$

[Compare M., p. 7.]

6. The intersection of two planes. Let the equations of two given distinct planes be

$$l_1 x + m_1 y + n_1 z + p_1 t = 0,$$
$$l_2 x + m_2 y + n_2 z + p_2 t = 0.$$

Since the planes are distinct, the coefficients $l_1, m_1, n_1, p_1; l_2, m_2, n_2, p_2$ are not proportional, and so we may assume, without loss of generality, that $l_1 p_2 - l_2 p_1$ does not vanish. By putting $y = 0$, and then solving the two equations, we see that the point

$$(n_1 p_2 - n_2 p_1, \quad 0, \quad p_1 l_2 - p_2 l_1, \quad l_1 n_2 - l_2 n_1)$$

lies in each plane; and, by putting $z = 0$, we see similarly that the point

$$(m_1 p_2 - m_2 p_1, \quad p_1 l_2 - p_2 l_1, \quad 0, \quad l_1 m_2 - l_2 m_1)$$

also lies in each plane. [The hypothesis $l_1 p_2 - l_2 p_1 \neq 0$ ensures that these coordinates do not all vanish.] Hence there are certainly two distinct points, which it will be convenient to call $P_1(x_1, y_1, z_1, t_1)$, $P_2(x_2, y_2, z_2, t_2)$, common to both planes. But then the coordinates of the point

$$P(\lambda x_1 + \mu x_2, \ \lambda y_1 + \mu y_2, \ \lambda z_1 + \mu z_2, \ \lambda t_1 + \mu t_2)$$

satisfy each of the two given equations, so that P lies in each of the given planes. Further, no point outside the line $P_1 P_2$ can be common to both planes; for if there were such a point, say P_3, then each of the given planes would coincide with the plane $P_1 P_2 P_3$, contrary to the hypothesis that they are distinct. Hence *the points common to two given planes lie in a straight line*. This line is called the *line of intersection* of the two planes.

If the two given planes are called π_1, π_2, then we denote their line of intersection by the symbol $(\pi_1 \pi_2)$.

A straight line is determined when two distinct planes through it are given, and so the coordinates of the points of a line satisfy each of *two* linear equations, say

$$l_1x+m_1y+n_1z+p_1t=0,$$
$$l_2x+m_2y+n_2z+p_2t=0.$$

These equations are called the *equations of the line*. The two equations are not themselves uniquely determined; either may be replaced by an equation formed by adding any multiple of the first equation to any multiple of the second. For example, the line whose equations are $x=t=0$ is equally well defined by the equations

$$ax+bt=0, \quad a'x+b't=0,$$

where $a/a' \neq b/b'$.

Theorem-examples. 1. The plane

$$\lambda(l_1x+m_1y+n_1z+p_1t)+\mu(l_2x+m_2y+n_2z+p_2t)=0$$

passes through the line of intersection of the two planes

$$l_ix+m_iy+n_iz+p_it=0 \quad (i=1,2).$$

Conversely, the equation of *any* plane through the line can be expressed in that form.

[Compare M., pp. 13–14.]

2. Two straight lines which lie in a plane have one point in common. If l_1, l_2 are the lines, we call the point (l_1l_2).

3. The line joining two points in a plane lies entirely in that plane. If A_1, A_2 are the points, we call the line A_1A_2.

4. A line not lying in a plane meets it in one, and only one, point. If l is the line and π the plane, we call the point $(l\pi)$, or (πl).

5. A unique plane can be drawn through a given point to contain a given line not passing through that point. If P is the point and l the line, we call the plane $[Pl]$, or $[lP]$.

6. A unique plane can be drawn to contain two given lines with one point in common. If l_1, l_2 are the lines, we call the plane $[l_1l_2]$.

7. Two lines which do not lie in a plane have no common point, and two lines which have no common point do not lie in a plane. Two such lines are said to be *skew*.

8. Three distinct planes either have a line in common or meet in a unique point. In the latter case, if π_1, π_2, π_3 are the planes, we call the point $(\pi_1\pi_2\pi_3)$.

9. Three lines, each of which meets the other two, either lie in a plane or meet in a point.

10. The four planes whose equations are

$$\begin{aligned} -ny + mz - l't &= 0, \\ nx - lz - m't &= 0, \\ -mx + ly - n't &= 0, \\ l'x + m'y + n'z &= 0, \end{aligned}$$

where the numbers l, m, n, l', m', n' are connected by the relation

$$ll' + mm' + nn' = 0,$$

have a line in common.

[Consider $n'(2) - m'(3) + l(4)$, etc.]

DEFINITION. The system of planes passing through a given line l is called a *pencil*. Thus if l is defined by the planes π_1, π_2 whose equations are

$$\pi_1 \equiv l_1 x + m_1 y + n_1 z + p_1 t = 0,$$
$$\pi_2 \equiv l_2 x + m_2 y + n_2 z + p_2 t = 0,$$

then the equation of any plane of the pencil is

$$\pi_1 + \lambda \pi_2 = 0.$$

The line l is called the *axis* of the pencil.

In the same way, the system of points lying on a given line l is called a *range*. Thus if l is defined by the points $P_1(x_1, y_1, z_1, t_1)$, $P_2(x_2, y_2, z_2, t_2)$, then the coordinates of any point of the range are

$$(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2).$$

The line l is called the *base* of the range.

7. The syzygy connecting the symbols of five given points. Let P_1, P_2, P_3, P_4, P_5 be five given points, of which no four are coplanar. We prove that *their symbols are necessarily connected by a syzygy, uniquely defined by* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5$, *namely*

$$\lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \lambda_3 \mathbf{P}_3 + \lambda_4 \mathbf{P}_4 + \lambda_5 \mathbf{P}_5 \equiv \mathbf{0}.$$

We have to prove that the four equations

$$\begin{aligned} \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 + \lambda_5 x_5 &= 0, \\ \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4 + \lambda_5 y_5 &= 0, \\ \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 + \lambda_5 z_5 &= 0, \\ \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3 + \lambda_4 t_4 + \lambda_5 t_5 &= 0 \end{aligned}$$

can be solved uniquely for the ratios $\lambda_1:\lambda_2:\lambda_3:\lambda_4:\lambda_5$. It is well known that, if D_i denotes the determinant

$$D_i \equiv \begin{vmatrix} x_j & x_k & x_l & x_m \\ y_j & y_k & y_l & y_m \\ z_j & z_k & z_l & z_m \\ t_j & t_k & t_l & t_m \end{vmatrix},$$

where i, j, k, l, m are the numbers 1, 2, 3, 4, 5 in cyclic order, then

$$\frac{\lambda_1}{D_1} = \frac{\lambda_2}{D_2} = \frac{\lambda_3}{D_3} = \frac{\lambda_4}{D_4} = \frac{\lambda_5}{D_5},$$

where the determinants D_i are not zero, since no three points are collinear or four points coplanar.* The syzygy is therefore uniquely determined.

Theorem-example. The symbol of a point P in general position in space can be expressed in terms of those of four given points X, Y, Z, T (not coplanar) by means of a syzygy

$$\mathbf{P} \equiv \lambda\mathbf{X} + \mu\mathbf{Y} + \nu\mathbf{Z} + \rho\mathbf{T}.$$

8. The transversal from a given point to two given skew lines. Let P be a given point and l_1, l_2 two given skew lines not through P. The planes $[Pl_1]$, $[Pl_2]$ are uniquely determined (§ 6, Theorem-example 5), and they meet in a line, necessarily through P. This line meets l_1, l_2, since it lies in the planes $[Pl_1]$, $[Pl_2]$ respectively. It is called the *transversal* from P to the given lines.

Alternatively, let P be the given point and A_1B_1, A_2B_2 the two given skew lines. The five symbols $\mathbf{P}$, $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$ are connected by a syzygy, say

$$\lambda\mathbf{P} + \mu_1\mathbf{A}_1 + \nu_1\mathbf{B}_1 + \mu_2\mathbf{A}_2 + \nu_2\mathbf{B}_2 \equiv \mathbf{0},$$

which we can arrange in the form

$$\lambda\mathbf{P} + (\mu_1\mathbf{A}_1 + \nu_1\mathbf{B}_1) + (\mu_2\mathbf{A}_2 + \nu_2\mathbf{B}_2) \equiv \mathbf{0}.$$

This relation shows that the points with symbols $\mathbf{P}$, $\mu_1\mathbf{A}_1 + \nu_1\mathbf{B}_1$, $\mu_2\mathbf{A}_2 + \nu_2\mathbf{B}_2$ are collinear, as required.

* If three of the points were collinear, then one of the columns would be the sum of appropriate multiples of two others.

Theorem-example. If P lies on l_1, there is an infinity of lines meeting l_1 and l_2.

[This is, of course, obvious; but the reader should consider what happens to the syzygy in the alternative proof.]

9. The tetrahedron of reference.

DEFINITION. The figure formed by four non-concurrent planes is called a *tetrahedron*. The planes are called the *faces* of the tetrahedron, the four points in which three of the planes meet are called the *vertices*, and the six lines in which the planes meet in pairs are called the *edges*. Two vertices lie on each edge.

In particular, the four points $X(1, 0, 0, 0)$, $Y(0, 1, 0, 0)$, $Z(0, 0, 1, 0)$, $T(0, 0, 0, 1)$ are the vertices of a tetrahedron called the *tetrahedron of reference*. The faces YZT, ZXT, XYT, XYZ are given by the equations

$$x = 0, \quad y = 0, \quad z = 0, \quad t = 0$$

respectively, and the edges YZ, ZX, XY, XT, YT, ZT are given by the pairs of equations

$$x = t = 0; \quad y = t = 0; \quad z = t = 0;$$

$$y = z = 0; \quad z = x = 0; \quad x = y = 0$$

respectively.

10. The unit point and the unit plane. If the tetrahedron of reference is given, we can refer to it a system of coordinates in which any assigned point U, not on a face of the tetrahedron, has coordinates (1, 1, 1, 1), as follows:

Suppose that, in *any* coordinate system with that tetrahedron of reference, the coordinates of U are $(\alpha, \beta, \gamma, \delta)$. We can, as it were, 're-name' the points of space, by means of a *transformation* from the system x, y, z, t to a system x', y', z', t' given by the relations

$$x' = x/\alpha, \quad y' = y/\beta, \quad z' = z/\gamma, \quad t' = t/\delta.$$

Then (i) the coordinates of every point of space are determined in terms of x', y', z', t'; (ii) the point $(\alpha, \beta, \gamma, \delta)$ becomes the point (1, 1, 1, 1); (iii) the tetrahedron of reference is unchanged. We have therefore found a system of coordinates in which U is the point (1, 1, 1, 1), called the *unit point* for that system of coordinates.