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Theory of Matroids

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THEORY OF MATROIDS

Edited by
Neil White
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Series Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive change of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the nonspecialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

Gian-Carlo Rota

Foreword

It is a rare event that mathematicians, cozily ensconced in the world of established theories, should extract, by dint of pioneering work, some new gem that later generations will spend decades polishing and refining. The hard-won theory of matroids is one such instance. Rich in connections with mathematics, pure and applied, deeply rooted in the utmost reaches of combinatorial thinking, strongly motivated by the toughest combinatorial problems of our day, this theory has emerged as the proving ground of the idea that combinatorics, too, can yield to the power of systematic thinking.

A superficial look at the theory of matroids might lead to the conclusion that it is largely an abstraction of linear algebra. It was noticed quite some time ago that the elementary theory of linear dependence can be developed from the MacLane-Steinitz exchange axiom. In fact, this abstraction was first exploited in the theory of transcendence degrees of fields. But such a conclusion would be unwarranted. What the theory of matroids provides is a variety of cryptomorphic axiomatic approaches, each of which corresponds to a genuinely new way of looking at linear algebra. Linear algebraists might not easily be led to these axiomatic approaches. The axiomatization of matroids by the notion of a minimal dependent set, for example, leads to the deeper matching theorems for sets of vectors. The axiomatization by the notion of rank leads to the classification of projectively invariant constructions of new matroids from old. New axiomatizations are still appearing. Matroid theory is unique in mathematics in the number and variety of its equivalent axiom systems; this accounts in part for the versatility and applicability of the subject.

The idea of geometric lattice arose from the combinatorial theory of matroids and now provides a framework for the invariant theory of sets of points. The abstraction of the coloring problem of graphs to arbitrary geometric lattices has provided a remarkable unification, now called the critical problem, of a variety of deep extremal set-theoretic problems. One of the seldom-stated motivations for much current work on matroid theory is in fact the idea that the critical problem may eventually be solved by joint action of invariant theory, extremal set theory, and finite geometry, perhaps with a touch of homological algebra. The closely related and now well developed theory of coordinatization of matroids is spearheading these developments.

Surely some of the most beautiful results of contemporary combinatorics are Tutte's theorems relating the coordinatizability of matroids to the absence of certain forbidden minors (called obstructions) in its geometric lattice. Tutte's theory culminates in his characterization of unimodular matroids, namely matroids coordinatizable over every field, by the absence of the Fano plane.

Matroids have proved to be an essential concept in discrete optimization. The greedy algorithm is the optimization-theoretic analog of the MacLane-Steinitz exchange property. From this elementary beginning, interest in matroids in this field has exploded: polymatroids, oriented matroids, greedoids, and submodular functions now abound in the literature of combinatorial optimization.

What will happen in the long run to the theory of matroids? We predict that it will soon feed back profoundly upon linear and multi-linear algebra, and, most of all, upon homological algebra, in at least two ways: firstly, by providing a rich problematique which these fields can test themselves upon, and secondly, by feeding its own techniques directly back onto homological algebra. Matroid theory reflects in an exemplary way the mathematical preoccupations of our day, such as the meeting of the cross-currents of pure and applied mathematics and the cutting across party lines of separate fields, while not losing sight of the concrete objectives of solving some of the longstanding problems of contemporary mathematics.

Gian-Carlo Rota

Preface

This book had its beginnings over a decade ago, as a simple rewriting of Crapo and Rota's preliminary edition of *Combinatorial Geometries*, to be accomplished by Crapo, Rota, and White. We soon realized that the subject had grown enough, even then, that a more comprehensive compendium would be of greater benefit. This led, in turn, to the idea of soliciting contributions from many of the workers in matroid theory. Consequently, this work has grown too lengthy to be contained in a single volume. This is but the first of a projected three-volume series, although we are giving separate titles to each of the volumes. We are planning to call the remaining volumes *Combinatorial Geometries*, and *Advances in Matroid Theory*.

This first volume is a primer in the basic axioms and constructions of matroids. It will prove useful as a text because exposition has been kept a prime consideration throughout. Proofs of theorems are often omitted, with references given to the original works, and exercises are included. This volume will also be useful as a reference work for matroid theorists, especially Brylawski's encyclopedic chapter "Constructions" and his cryptomorphism appendix.

The volume starts with Crapo's chapter "Examples and Basic Concepts." This chapter is a very informal introduction to matroids, with lots of examples, that provides an overview of the subject. The next chapter is "Axiom Systems," by Nicoletti and White. This gets into the necessary work of proving the equivalence of some of the major axiom systems, a chore made easier by keeping in mind the familiar analogous concepts from linear algebra. The presentation in this chapter is based on Nicoletti's interesting self-dual metasystem of the axiom systems. In the third chapter, Faigle presents the lattice-theoretic approach to matroids. Originated by Birkhoff, this was the cornerstone of the original Crapo-Rota volume, although it is less heavily relied on in the current work. Then Kung provides more depth on one family of related axiom systems, the basis axioms. In the fifth chapter, Crapo explains the fundamental duality concept, which generalizes duality of planar graphs (a prime motivation in Whitney's creation of the concept of matroid) and orthogonality of vector subspaces. Next comes Oxley's exposition of one of the most important and most elementary classes of examples of matroids, those that arise from a graph. Included are a complete characterization of when two graphs produce the same matroid and a description of an important class of graphs that are characterized matroidally, the series-parallel networks. The seventh chapter is Brylawski's comprehensive compendium of matroid constructions, together with a marvelous separate index for this chapter only. This chapter includes new material never before published, most notably the idea of matroid bracing. Kung's "Strong Maps" is the first of two chapters on mappings between matroids. Strong maps have the beautiful characterization of always being factorable into an embedding followed by a contraction. Kung and Nguyen then present a much more general type of mapping, the weak map, which formalizes the intuitive notion that one matroid is "more dependent" than another. Nguyen then presents the theory of semimodular functions, another approach to matroids using their rank (or dimension) functions. Finally comes Brylawski's "Appendix of Matroid Cryptomorphisms," a detailed listing of all of the known equivalent axiomatizations of matroids, the cryptomorphisms that relate these axiomatizations, and their applications to the most important classes of examples of matroids. A previous version of this listing has been available to matroid theorists for some years and has proved to be a useful reference work.

Exercises marked with an asterisk tend to be more difficult, and those with two asterisks are unsolved. In order to make it easier to distinguish in the figures between affine diagrams of matroids and other representations, such as lattice diagrams and graphs, we have adopted a new convention of using large dots for elements in affine diagrams, and small dots for lattice elements and vertices of graphs.

I would like to thank all the contributors to this project for their continuing support, despite the delays in the appearance of this volume. I would also like to thank Henry Crapo and Gian-Carlo Rota for involving me in this project and for their support. Thanks are due as well to several outside referees, who must remain unnamed. I also thank Rhodes Peele for working many of the exercises before their inclusion. Finally, Tom Brylawski wishes to thank the National Science Foundation for their partial support of his work under grant MCS 7801149, and Hazeline Lewis and Daniela Calvetti for their assistance in preparing his manuscript.

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