CHAPTER 1

Examples and Basic Concepts

Henry Crapo

1.1. EXAMPLES FROM LINEAR ALGEBRA AND PROJECTIVE GEOMETRY

As an introduction to the concepts of *combinatorial geometry* and *matroid*, we wish to emphasize those features of the theory that have given it a unifying role in other branches of mathematics, that have permitted it to be fruitfully applied in disparate domains of science, and that continue to arouse broader interest in the subject. It is our intention in this chapter to clarify the basic concepts by showing how they appear and are interrelated in a list of significant examples. This will give the reader a general orientation with respect to the basic concepts, prior to their axiomatic treatment in Chapter 2. Most of these examples will be dealt with in full detail in subsequent chapters.

The concept of *combinatorial geometry* arose from work in projective geometry and linear algebra. The focus of this work was to understand the basic properties of two relations:

(1) the *incidence* between points, lines, planes, and so on (which in general we call *flats*) in geometries and geometric configurations
(2) the *linear dependence* of sets of vectors.

The task to characterize (axiomatize) these relations of incidence and linear dependence seems in retrospect both urgent and feasible in the light of far-reaching applications, both to new geometries and to more general algebraic and combinatorial structures.

Research in combinatorial geometry has been concentrated on

(1) *synthetic* (*combinatorial*) *methods*, involving only *incidence relations* between flats, and the fundamental operations of *projection* and *intersection*
(2) *intrinsic properties* of configurations, internal properties that configurations possess independent of the way in which they may be represented or constructed within some conventional space.

This emphasis on synthetic methods and on configurations has permitted the theory and its applications to develop without undue reliance on coordinate systems, vector algebra, determinants, and the like. In the resulting combinatorial theory, theorems are often phrased in terms of the existence of certain geometric incidences or subconfigurations. Premises and conclusions of this nature are often more practical to use in applications, and are then a greater support to intuition, than their equivalent algebraic formulations. But there is a price to pay: There exist combinatorial geometries whose points cannot be coordinatized in any satisfactory manner.

1.1.A. Subsets of Projective Spaces

The most fundamental example of a combinatorial geometry is the structure of an arbitrary set of points in a finite-dimensional projective space. To begin with, the projective space has a structure of *flats*, namely its points, lines, planes, and so on, and the relation of incidence between them. Each flat has a *rank* equal to its projective dimension plus 1; points have rank 1, lines have rank 2, and so forth. The rank of any flat is thus equal to the number of points needed to determine that flat.

Our immediate task is to see how this structure of flats can be induced on a *subset* of the set of points of the space. Let $E \subseteq S$ be a subset of the set S of points in some projective space. Each projective flat itself consists of a set of points, whose intersection with E we call a *flat* of the combinatorial geometry $G(E)$ induced on the set E. How should we assign a rank to such a flat? Say a subset $A \subseteq E$ is the intersection with E of some projective flat. Then there is a *least* such projective flat whose intersection with E is A, and it has a rank that we assign as the rank $r(A)$ of the flat A in the geometry $G(E)$.

In Figure 1.1a we sketch a combinatorial geometry with six points. The flats and their relations of incidence are indicated in the lattice diagram of Figure 1.1b. Straight lines in the sketch indicate those lines of the geometry that contain more than two points, but pairs like dg are also lines of the combinatorial geometry, and have rank 2.

This geometry is easily shown to be the geometry induced on a set of six points in the real projective plane. In Figure 1.1c we indicate one such choice of six points. Here, for instance, the collinearity of points g, l, and f

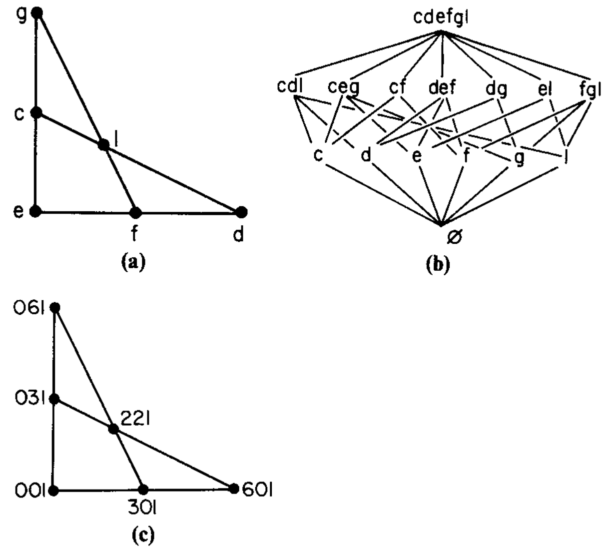


FIGURE 1.1. A six-point geometry, presented in three ways.

is proved by the computation

$$-g + 3l = -1(0,6,1) + 3(2,2,1) = (6,0,2) \simeq (3,0,1) = f,$$

because coordinates differing only by a common scalar multiple refer to the same projective point.

The lattice of flats of a combinatorial geometry, such as that in Figure 1.1b, is a *geometric lattice*. The *join* of two flats is the least flat containing both of them; the *meet* of two flats is their intersection. The main properties of geometric lattices are summed up in the following statement: If a flat A is contained in a flat B, then they are in consecutive ranks if and only if there is a point $p \notin A$ such that $A \vee p = B$. Put another way, the set of points *not* on any fixed flat A is partitioned by inclusion of those points in the flats of the next higher rank, containing A. Thus, in a combinatorial geometry of rank ≥ 3, the points not on a given line are partitioned by inclusion in the planes passing through that line. For instance, in the projective cube of Figure 1.2a, the set *abcefh* of points not on the line *dg* is partitioned $(af)(b)(ch)(e)$ by inclusion in the planes *adfg*, *bdg*, *cdgh*, and *deg*, respectively.

We wish to hold off axiomatic treatment of combinatorial geometries until the next chapter; here we shall deal further with examples and sketch the general outlines of the basic theory. For the present, let us agree that a combinatorial geometry consists of *flats* with a well-defined *rank*, each flat being a set of *points*, with the following property:

(1) The set of points *not* on any fixed flat A is partitioned by inclusion in the flats of the next higher rank, containing A.

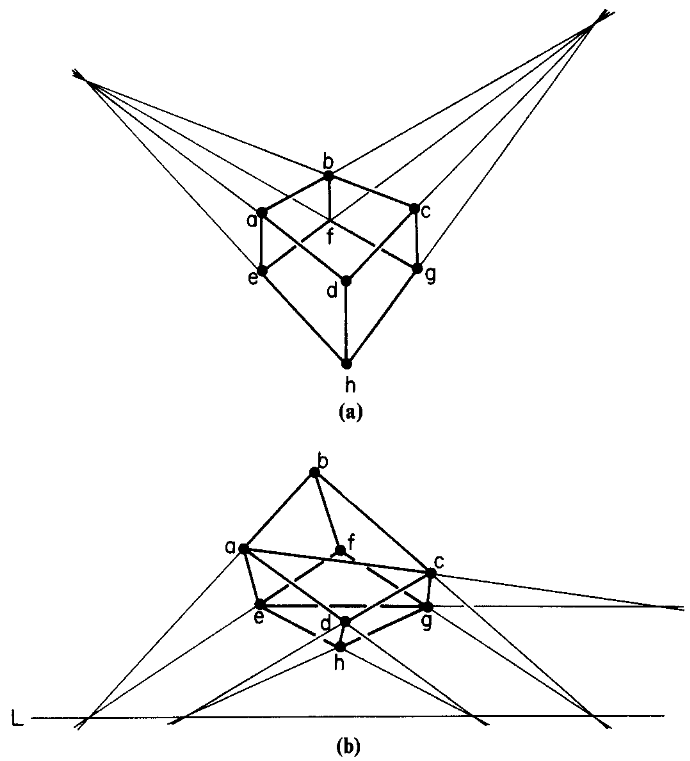


FIGURE 1.2. (a) A projective cube. (b) A noncoordinatizable geometry.

That statement, together with the assertions that

(2) any intersection of flats is a flat,
(3) rank agrees with height in the lattice,
(4) the overall geometry has finite rank, and
(5) the empty set and single points are flats,

already constitutes more than enough axioms to characterize combinatorial geometries (see Exercise 2.7).

There are significant differences between combinatorial geometries in general and projective geometries in particular. In any combinatorial geometry, concurrent lines must be coplanar, and planes meeting in a line must be cospatial. In a projective geometry, we also know that coplanar lines are concurrent, and cospatial planes must meet in a line. This is not true for combinatorial geometries in general. For instance, the coplanar lines *cf* and *el* do not meet in the combinatorial geometry of Figure 1.1a. If Figure 1.2a is construed as a combinatorial geometry, the cospatial planes *aceg* and *bdfh* do not meet at all!

A slight modification of the geometry of a set of points in projective space can produce a geometry that cannot be realized in any projective space. Consider Figure 1.2b, which is intended to suggest a projective cube with six flat faces, in which the diagonal set of four points *aceg* is coplanar, but the diagonal set *bdfh* is skew. If eight points lying in this way on seven four-point planes were to be found in a projective geometry, the following contradiction would arise: The planes *bafe*, *aceg*, and *cbgf* would meet in some point *p*, a point that would thus lie on the three lines *ae*, *cg*, and *bf* of intersection of those planes. Similarly, the lines *ae*, *cg*, and *dh* would be concurrent and could meet only at the same point *p*. Thus, *bf* and *dh* would be concurrent at *p* and would *have to be coplanar*, rather than skew. The drawing is designed to make it seem reasonable that the lines *bf* and *dh* could be skew. Actually, in any three-dimensional realization of this particular plane drawing, the lines *ac* and *eg* are also skew, and neither of the diagonal sets *aceg* and *bdfh* is coplanar. If the set *aceg* is to be coplanar, the lines *ac* and *eg* must meet on the line *L*, to provide a point of intersection for the plane *aceg* with the top and bottom planes *abcd* and *efgh* of the projective cube.

From the foregoing discussion we can see that it is a theorem of projective geometry that if one of these diagonal sets is coplanar, so is the other. However, a set of eight points in rank 4, no three points collinear, forming 7 four-point planes *abcd*, *abef*, *aceg*, *adeh*, *bcfg*, *cdgh*, and *efgh* and 28 three-point planes *does form a combinatorial geometry*. We say the geometry is *not projectively coordinatizable.*

It would not be correct to imagine that all finite-point sets in two- or three-dimensional projective space can be illustrated by drawings such as those of Figures 1.1 and 1.2. When points are chosen from projective spaces over finite fields, we are often forced to represent the major lines and planes as *curved*. Figure 1.3b shows the entire projective plane over the field $GF(3) = \{0, 1, 2\}$ mod 3. To prove that this geometry is the projective plane over GF(3), assign coordinates to each of the points in a basis, say $a = (0, 0, 1)$, $b = (0, 1, 0)$, $d = (1, 0, 0)$, $e = (1, 1, 1)$. Because the point i is on the line *ae*, and every line has exactly four points over GF(3), the coordinates of the point i must be either $(1, 1, 0)$ or $(1, 1, 2)$ modulo 3. Because the point i is also on the line *bd*, the correct choice is $(1, 1, 0)$, and the fourth points on those two lines, namely l and m, must be $(1, 1, 2)$ and $(1, 2, 0)$, respectively. Continuing in this way, we can easily compute a complete set of coordinates for the geometry and thus verify that it is isomorphic to the stated projective plane. In this case, $k = (0, 1, 1)$, $c = (0, 2, 1)$, $f = (2, 1, 1)$, $j = (1, 0, 1)$, $g = (1, 0, 2)$, and $h = (1, 2, 1)$.

If we select one hyperplane in a projective geometry to be the "hyperplane at infinity", the "finite" points not on that hyperplane form an *affine geometry*, a subgeometry of the projective space. Figure 1.3c shows such an affine geometry, obtained by removing the line *jklm* from the projective geometry in Figure 1.3b.

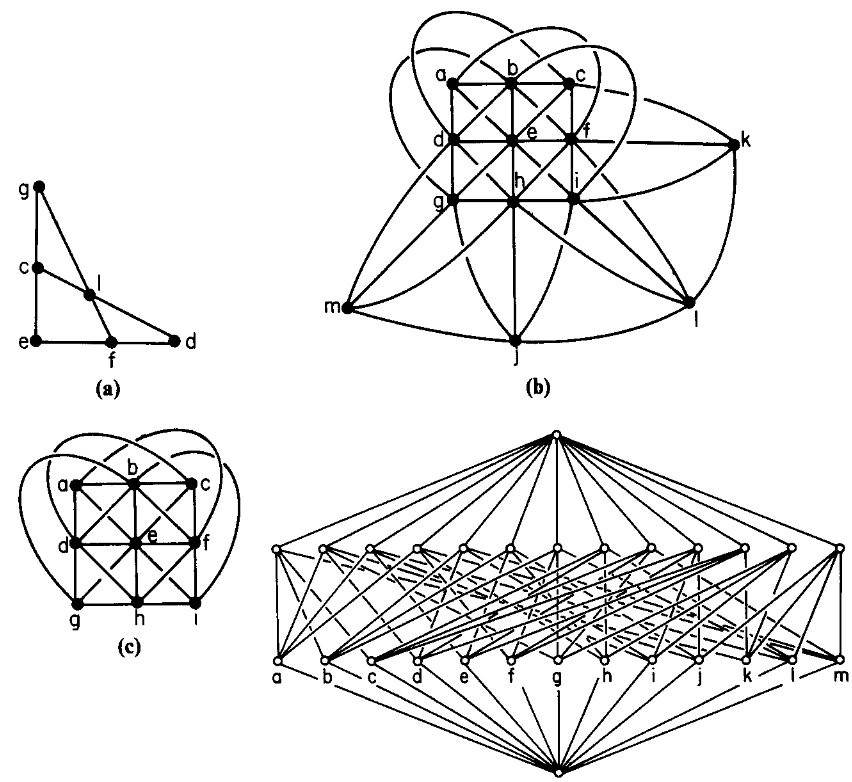


FIGURE 1.3. (a) A six-point geometry. (b) The projective plane over GF(3). (c) The affine plane over GF(3). (d) The lattice of flats of the projective plane over GF(3).

The geometry in Figure 1.1a (= Figure 1.3a) is also a subgeometry of the projective plane over GF(3). The choice of lettters as labels for its points was made in such a way as to show this embedding: Observe that the rank of any subset of the set *cdefgl* of points is the same when measured in the subgeometry or in the whole projective geometry *ab . . . m*.

The lattice of flats of the projective plane over GF(3) is drawn in Figure 1.3d. It has a property not shared by all geometric lattices. The reverse-order, or *opposite*, lattice is also geometric. Thus, if two flats of rank k have a join of rank $k + 1$, their intersection will have rank $k - 1$. In particular, any two distinct hyperplanes *cover* their intersection. Such geometric lattices are *modular*. Subgeometries (formed by selecting a subset of the set of points, and taking all joins of those elements) of modular geometries do not need to be modular. Note how, for instance, the lines *le* and *cf* meet at the point i in Figure 1.3d (or 1.3b), but have no common point in the subgeometry, Figure 1.3a. The "parallel" lines ae and dh of the affine configuration Figure 1.3c meet at the point l on the "line at infinity" selected in Figure 1.3b.

The operation of *projection*, central to the theory of projective geometry, involves two lattice operations. If a geometry G is to be projected from a center C onto a screen S, we use the lattice operation *join* to create from each flat A of G the flat $A \vee C$ (the bundle of *rays* of the projection) and then use the lattice operation *meet* to find the image $(A \vee C) \wedge S$ where these rays meet the screen. The first of these two operations is a natural operation on geometric lattices, and creates a *quotient* structure whose flats are certain of the flats of G. For instance, if the subgeometry in Figure 1.3a is projected from the point j in the projective plane (Figure 1.3b) as center, the distinct joins of flats of G with j are j itself, the four lines through j, and the entire plane. The sets of points of G that lie in these various flats are the empty set, l, cf, e, dg, and the entire set. It is this set of flats that forms the *quotient* of G associated with this projection. Geometrically, it is a four-point line in which two of the points are represented by pairs of elements.

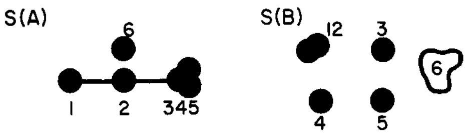
This is where the distinction between matroids and combinatorial geometries becomes significant. The quotient structure arising from a projection is not, strictly speaking, a geometry. We must either say that the quotient geometry is in this case based on a four-element set of points or else introduce a broader concept that will admit multiple points and even "zero" points that are in the geometric closure of the empty set. This broader concept is that of a *matroid*. The quotient matroid of the projection under discussion has no zero elements, but if we were to project the same subgeometry from one of its own points as center, that point would be a zero element (or *loop*) of the quotient matroid.

1.1.B. Vector Geometries

Many of the basic concepts of combinatorial geometry are straightforward generalizations of concepts in linear algebra. Given a set E of vectors in a finite-dimensional vector space V, every subset A of E will have a well-defined rank $r(A)$ equal to the dimension of the vector space spanned by the vectors in A. Those subsets $A \subseteq E$ that are intersections with E of subspaces of V are the flats of a matroid. We call such a structure a *vector matroid* (or *vector geometry*).

The passage from a vector matroid to its associated combinatorial geometry is the familiar passage from vectors to points in projective space: The zero vector remains in the closure of the empty set, and vectors that are scalar multiples of one another yield the same projective point. (Think of projection of Euclidean n-dimensional space from the point with position vector zero as center; the quotient of this projection is a vector matroid.)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$



$$A' = \begin{bmatrix} 0 & 1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 2 & 2 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

FIGURE 1.4. A vector matroid and its orthogonal matroid.

The notions of *bases* and *dependent* and *independent sets* arise from the example of vector geometries. A set of vectors is linearly dependent if and only if some nontrivial linear combination of them is equal to the zero vector. Although the values of the scalars used in linear combinations have no combinatorial meaning, the set of all dependent sets in a given vector geometry carries enough information to characterize the geometry up to combinatorial isomorphism. The minimal dependent sets, which we call *circuits*, are also a starting point for an axiomatization of combinatorial geometries.

Matrices provide particularly interesting examples of vector geometries. The rows of an m-by-n matrix (rank k) yield a vector matroid on m elements, rank k, and the columns yield a vector matroid on n elements, also of rank k. The matrix A in Figure 1.4, for instance, has rank 3. The three rows being independent, the corresponding geometry consists of three points not collinear, forming a triangle. Now consider the geometry of columns of this matrix A. Columns 3, 4, and 5 are multiples of one another, being equal; so they represent the same projective point. The first three columns form a singular 3-by-3 matrix, because there is a repeated row. So columns 1, 2, and 3 are dependent, and represent collinear points. The sixth column is independent of the other five; so the vector geometry is that labeled $S(A)$ in Figure 1.4.

It is easy to recognize the *bases* (maximal independent sets) of such a vector geometry. A basis must *span* the geometry, and must contain no circuits, that is, no zero element, no pair of equal points, no three collinear points, no four coplanar points, and so forth. In the geometry of columns of the matrix A, 126, 136, 146, 156, 236, 246, and 256 are the only bases.

The six columns of the matrix A have many dependences. The coefficients of all minimal dependences (circuits) form the rows of the matrix B. There are only six circuits, three consisting of pairs 34, 35, and 45 of equal points, the other three consisting of triples 123, 124, and 125 of collinear points. No circuit contains the element 6. Note that every row of the matrix B is orthogonal to (has inner product zero with) every row of A, and the rows of A and B together span the entire six-dimensional vector space. Thus, the row spaces of the matrices A and B are orthogonal complements of one another.

The columns of the matrix B form a geometry $S(B)$ that we call the *orthogonal* $S^*(A)$ of the geometry of columns of A. The combinatorial relations between these orthogonal geometries $S(A)$ and $S(B)$ can be summarized briefly as follows:

(1) The complement of a basis for $S(A)$ is a basis for $S(B)$, and conversely,
(2) For any element p and any bipartition of the remaining elements into two sets X and Y, the element p depends on the set X in the geometry $S(A)$ if and only if p does not depend on the set Y in $S(B)$.

Let us take a close look at the geometry of columns of the matrix B. The sixth column being the zero vector, it is an element in the closure of the empty set and does not represent a point of the geometry. Columns 1 and 2 represent the same point. Columns 2, 3, 4, and 5 form a circuit, a dependence being given by the second row of the matrix A. Thus, the geometry of columns of B consists of four coplanar points 12, 3, 4, and 5 and a "phantom" element 6. Once the geometries $S(A)$ and $S(B)$ are drawn, we can check the foregoing orthogonality relations (1) and (2). Bases for the geometry $S(B)$ consist of three noncollinear points. There are seven such three-element sets, 345, 245, . . . , and these are precisely the complements of the seven bases listed earlier for the geometry $S(A)$. Checking an example of property (2) of orthogonal geometries, notice that point 2 is in the closure of the set 345 in the geometry $S(B)$; so the point 2 is *not* in the closure of the complementary set 16 in the geometry $S(A)$.

Finally, look at the rows of the matrix B. These correspond to the circuits of the column geometry $S(A)$ and can be regarded as forming a geometry in their own right, the *geometry of circuits* of $S(A)$. This is not a combinatorial construction: Changes in the coordinatization of a geometry can change the linear relations between its circuits, without changing the circuits themselves (as subsets). Its combinatorial interest arises from the following fact: If we draw the lattice for the row geometry of the matrix B, we find an embedded copy of the lattice for $S(B)$, the column geometry of B, within it in inverted order (Figure 1.5). We say that the row geometry $R(B)$ is an *adjoint* of the column geometry $S(B)$.

Now let us take a second look at the two matrices in Figure 1.4. There is a certain lack of symmetry in this example. Although the rows of A are orthogonal to the rows of B, the rows of B have the special property that they come from the circuits of the column geometry $S(A)$; they are (up to a scalar factor in each vector) all the *minimal support vectors* in the space

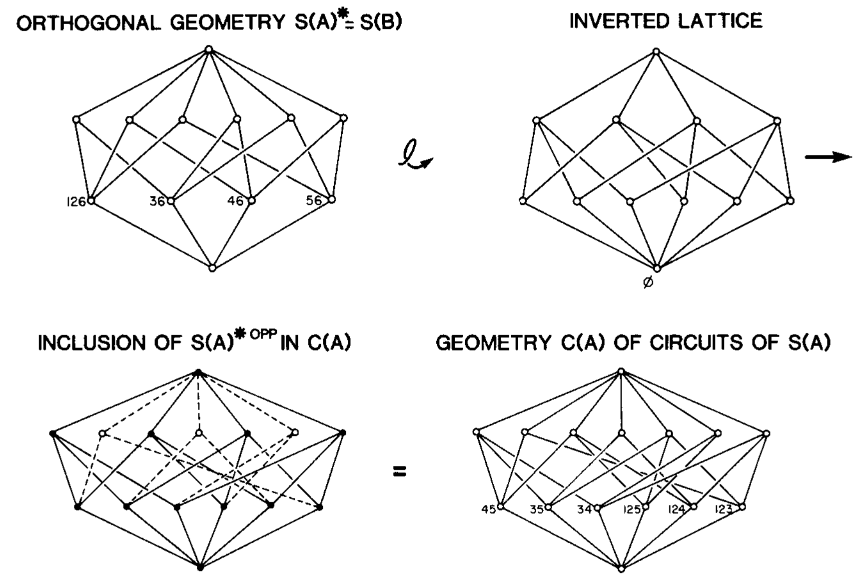


FIGURE 1.5. A lattice inclusion.

$R(A)^{\perp}$, the kernel of the linear transformation "left-multiplication by A." A similar construction would replace the matrix A by a matrix A' (Figure 1.4) whose rows are the minimal support vectors in the space $R(B)^{\perp}$, that is, by all minimal support vectors in the row space $R(A)$. The support sets of these minimal support vectors are combinatorially significant. The complements of these sets are the hyperplanes (copoints) of the column geometry $S(A)$. Thus, in the given example, the four lines 16, 26, 3456, and 12345 have as complements the minimal support sets 2345, 1345, 12, and 6, respectively.

These minimal support sets are called the *bonds* of the matroid $S(A)$. (The term *bond* had its origin in another context, namely in the theory of graphs, where the term refers to a link between two parts of a graph, as will be explained later.) *Bond* and *circuit* are dual notions, in the sense that the bonds of any matroid are the circuits of its orthogonal matroid.

The basic fact to remember is that the zero set of any vector in the row space of any matrix A is a flat in the column geometry $S(A)$, and the zero set of any vector orthogonal to all the rows is a flat of the orthogonal geometry. Furthermore, passage from a matrix A to the matrix A', whose rows are the minimal support vectors of the row space $R(A)$, does not change the column geometry $S(A) = S(A')$.

1.1.C. Function-Space Geometries

A *function-space geometry* (or *chain-group geometry*) $G(X, V)$ is a geometry on a set X of points, whose flats are determined as follows by a finite-dimensional vector space V of functions from a set X into a field F. A subset