Introduction

It is legitimate to consider J. Fourier (1768–1830) as the initiator of the theory of integral equations, owing to the fact that he obtained the inversion formula for what we now call the 'Fourier transform': under adequate restrictions on the functions f and g, from $g(x) = (2\pi)^{-1/2} \int \exp(ixy) f(y)\, dy$ one derives $f(y) = (2\pi)^{-1/2} \int \exp(-ixy) g(x)\, dx$. Of course, one can interpret the inversion formula as providing the inverse operator (an integral operator!) of the Fourier integral operator. This interpretation was adopted towards the end of the last century by V. Volterra, who identified the problem of solving integral equations with the problem of finding inverses of certain integral operators.

Abel [1], [2] dealt with the integral equation known as 'Abel's equation', namely

$$\int_0^x (x-t)^{-\alpha} u(t)\, dt = F(x), \quad 0 < \alpha < 1,$$

where $u(t)$ stands for the unknown, while $F(x)$ is an assigned function. The solution of Abel's equation was provided by the formula

$$u(t) = -\pi^{-1} \sin \alpha\pi \frac{d}{dt} \int_0^t (t-x)^{\alpha-1} F(x)\, dx.$$

There are two interesting features related to Abel's equation. First, the so-called kernel $k(t) = t^{-\alpha}$ has a singularity at the origin, and second, Abel's equation is a convolution type equation. Both features have had a significant impact on the development of the theory of integral equations. It is interesting to point out that Abel found his integral equation in the case $\alpha = \frac{1}{2}$, starting from the following mechanical problem: find a curve C lying in a vertical plane xOy, such that a material point sliding without friction on C, and starting with zero initial velocity at a given point P_0 on C, reaches the origin O on C in an interval of time which is a given function of the y-coordinate of P_0. The method of solution used by Abel was neither simple nor rigorous, as pointed out by Schlömilch [1]. Schlömilch proposed a more rigorous approach, based on the reduction of multiple integrals to simpler ones.

Abel's work on integral equations exerted a tremendous influence during the following century. In 1860, Rouché [1], in 1884 Sonine [1], in 1888 du Bois-Reymond [1] were dedicating their efforts to the solution of various integral equations, particularly Abel's equation. du Bois-Reymond is credited with the introduction of the term 'integral equation'. In 1895, Levi-Civita [1] generalized the results obtained earlier by N. Sonine, still being concerned with Abel's type equations. Goursat [1] dealt with the solution of Abel's type equations, regarded as a problem of inversion of an integral, and in 1909 Myller [1] dealt with some problems in mechanics by using Abel's type equations as well as Fredholm's type equations (see Section 1.1). One century after Abel introduced and studied the integral equations now known as Abel's equations, mathematicians like Tamarkin [1] and Tonelli [1] were dedicating their attention to this subject.

But the year 1895 marked a new beginning in the theory of integral equations, due mainly to Volterra [2]. Unlike most of his predecessors, who were aiming at finding the solution of the equation by means of formulas or who dealt with special cases of what we now call 'Volterra equations' (the term was introduced by T. Lalesco who wrote his thesis on this topic with E. Picard), Volterra considered more general equations such as

$$\int_0^x k(x,t)u(t)\,\mathrm{d}t = F(x),$$

or

$$u(x) + \int_0^x k(x,t)u(t)\,\mathrm{d}t = F(x),$$

in which u stands for the unknown function. Volterra, who is also one of the founders of functional analysis, regarded the problem from a functional analytic standpoint, his main concern being the proof of the existence of the inverse of an integral operator. Volterra himself was not at first concerned with the various physical applications of the concept of an integral equation, the preoccupations which played a preponderant role later in his activities. In the same year 1895 when Volterra began his fundamental contribution to the theory of integral equations, Le Roux [1] published a notable paper in which integral equations appear as a powerful tool in investigating partial differential equations.

It is interesting to note that more than a half century before Volterra produced his contributions, Liouville [3] discovered the fact that the differential equation $y'' + [\rho^2 - a(x)]y = 0$, under initial conditions $y(0) = 1$, $y'(0) = 0$, is equivalent to the integral equation (of Volterra type and second kind)

$$y(x) - \rho^{-1}\int_0^x a(t)\sin\rho(x-t)y(t)\,\mathrm{d}t = \cos\rho x.$$

Also, apparently unaware of Abel's investigations concerning integral equations, Liouville [1], [2] dealt with the singular integral equation

$$\int_x^\infty (s-x)^{-1}y(s)\,\mathrm{d}s = f(x),$$

for which the solution is given by the formula

$$y(x) = -\pi^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty (s-x)^{-1/2}f(s)\,\mathrm{d}s.$$

Five years after Volterra made his first famous contribution to the theory of integral equations, Fredholm [1], [2] built up a new theory for integral equations containing a parameter of the form

$$y(x) + \lambda\int_a^b k(x,t)y(t)\,\mathrm{d}t = f(x).$$

Fredholm's theory, basically representing the extension of solvability of a linear finite-dimensional system $x + \lambda Ax = b$ to the infinite-dimensional case, has constituted one of the most valuable sources for the foundation of what is now known as linear functional analysis. It is instructive to notice that the fundamental paper by Banach [1] provides in its title the motivation for the necessity of new mathematical structures (Banach spaces), while Riesz (see Riesz and Sz.-Nagy [1]) proceeds to the generalization of Fredholm's theory to the case of abstract operators in linear spaces. In the years following the publication of Fredholm's results, illustrious mathematicians of the beginning of this century made their contributions to this subject: Poincaré [1], [2], Hilbert [1], Picard [1], Weyl [1], Fréchet (see Heywood and Fréchet [1]) and Schmidt [1]. The theory was considerably enriched and developed, and its connections to other branches of science were emphasized. In particular, mainly owing to Hilbert and Schmidt, the theory of equations with symmetric kernel was substantially developed, and important results about the orthogonal series were obtained.

Owing to the tremendous success that Fredholm's theory enjoyed during the first decades of this century, the theory of Volterra equations remained somewhat in its shadow. Most books on integral equations written during that period, as well as in more recent years, dismiss Volterra's equation by noticing that the corresponding integral operator has only the eigenvalue zero, and is therefore uninteresting. This approach is not very consistent because: first, if we deal with Volterra equations on infinite intervals, then the difference between them and Fredholm's equations is no longer so striking (with respect to the spectral properties); second, for Volterra equations it is possible to develop a theory of local existence, making these equations very useful in modelling various phenomena from applied fields of science; third, with the advance of nonlinear functional analysis during the past three decades, it has been possible to develop the study of Volterra nonlinear equations to a level which has not yet been reached by the theory of Fredholm nonlinear equations (both in terms of the number of results obtained by the researchers involved in this field, and the variety of the applications). We hope that this feature is adequately emphasized in this book.

Volterra was deeply preoccupied with the possible applications of the theory of integral equations in other fields of science. Among the most interesting applications found for the Volterra equations, we should first mention the so-called 'hereditary mechanics', also known as the 'mechanics of materials with memory'. The roots of this theory can be traced back to Boltzmann [1]. But Volterra advanced a rather sophisticated theory in the 1920s (see Volterra [2]), a theory which underwent some modification during the last three decades and which we can consider as still developing. In his studies during the 1920s on hereditary mechanics, Volterra was led to equations with infinite retardation (delay). Realizing that the mathematical apparatus was not yet developed at that time for undertaking successfully such investigations, Volterra 'cut' the delay, transforming his equations into equations with finite delay. Unfortunately, at that time, the theory of equations with finite delay was practically nonexistent, so that real progress had to be postponed.

Another field in which the theory of integral equations of Volterra type has found significant applications, beginning in the 1920s, is population dynamics. In Volterra and d'Ancona [1] one finds a synthesis of the first generation research pertaining to this field. This kind of study has been developed in the last two decades, and further progress is to be expected. A recent monograph on these matters is that of Webb [3]. See also Cushing [3].

Let us also mention the applications that the Volterra equations have found in Economic theory. An adequate reference in this respect is Samuelson [1].

In 1929, Tonelli [2] introduced the concept of a Volterra operator, as an operator U acting between function spaces, such that $x(t) = y(t)$ for $t \leq T$ implies $(Ux)(T) = (Uy)(T)$. Such operators are also known as *causal* operators or *non-anticipative* operators. Since most phenomena investigated by means of mathematical models are causal phenomena, the importance of this class of operators is obvious. Followers of Tonelli, such as Graffi [1] and Cinquini [1], produced some of the first contributions towards the foundations of the theory of abstract Volterra operators. In 1938, Tychonoff [2] emphasized again the significance of the theory of abstract Volterra operators in connection with the numerous applications in mathematical physics. He advanced this theory and introduced a more modern approach which contributed to its development, and served as a model for future investigations. The theory of abstract Volterra operators and equations has made substantial progress during the last 10–15 years, as will be illustrated in this book (see Chapters 2 and 3, particularly). The further development of this theory is one of the major problems in the theory of integral equations, viewed as an extension of the classical theory. Without such development, it seems unsatisfactory to pretend that the mathematical tools available for the investigation of phenomena in which heredity (hysteresis) occurs are powerful enough.

Before we move on to the examination of the theory of integral equations in more recent times, we take this opportunity to mention the contributions of Carleman [1] from the 1920s, and von Neumann [1] from the 1930s. Carleman's contribution should be regarded as the beginning of the theory of unbounded linear integral operators. This theory is still under development, and fruits are likely to follow. A recent monograph on Carleman's operators is due to Korotkov [1]. The spectral theory of integral operators is another story which holds great promise for the future. Recent noteworthy contributions are due to Pietsch [1], [2] and to Elstner and Pietsch [1].

During the 1940s there was little progress in the theory of integral equations. One remarkable contribution was due to Dolph [1], who made a substantial addition to the nonlinear theory of Hammerstein equations. The interaction between the nonlinearity of the equation and the spectrum of the linear integral operator involved in the equation is illustrated in a simple manner. Another basic contribution worthy of mention was due to Akhiezer [1] and relates to the theory of Carleman operators.

The 1950s were more eventful with regard to the theory of integral equations. In 1953, Sato [1] dealt with Volterra nonlinear equations from the standpoint of qualitative theory. In 1956 the Russian edition of the book by Krasnoselskii [1] was published. In the late 1950s M. G. Krein and I. C. Gohberg started a series of research efforts directed towards the build-up of a theory for classes of convolution equations on a half-axis, or on the entire real axis. Resolvent kernels were constructed and the behavior of solutions was investigated in a systematic manner for equations that, in general, possess a continuous spectrum. An account of this theory is given in my book (C. Corduneanu [4], Ch. 4, Wiener–Hopf equations). More developments of this theory are included in the monograph by Gohberg and Feldman [1]. More recent results in this direction can be found in Gohberg and Kaashoek [1]. Also in the late 1950s, the first work by V. M. Popov was published relating to the use of integral equations occuring in the theory of feedback systems. These are Volterra equations of convolution type, containing one or more nonlinearities. Fundamental stability results were obtained within a short time by Popov and many of his followers. In my book (C. Corduneanu [4], Ch. 3), most of the results obtained before 1970 in this direction have been included or reviewed. The research work in this field has been continued by many authors (particularly in Soviet Union and Romania). As recently as 1976, Nohel and Shea [1] published results in this line (frequency domain criteria of stability or other kinds of asymptotic behavior of solutions).

From the 1960s onwards, interest in the theory of integral equations has reached a level of concentration unknown since the years following Fredholm's investigations of this topic. Many research schools in the United States, Soviet Union, Italy, India, Japan, Finland, Romania, Poland, Israel, and other countries are directing their efforts towards the investigation of various problems related to the theory of integral operators and integral equations. While this growing interest is motivated in part by the numerous applications that integral equations have found in the mathematical modeling of phenomena and processes occuring in various areas of contemporary research, it should also be stressed that the development of the methods in nonlinear analysis has made possible the successful investigation of this kind of problem. There are presently several journals dedicated to the theory of integral equations and operators: the *Journal of Integral Equations and Applications* (the new series being published by Rocky Mountains Mathematical Consortium), *Integral Equations and Operator Theory* (Birkhauser), as well as the semiperiodic publication *Investigations on Integro-differential Equations in Kirkizia* (Russian). Journals such as *Differential Equations* (transl.), the *Journal of Mathematical Analysis and Applications*, and the *Journal of Differential Equations* contain numerous contributions dedicated to the theory of integral equations. Mathematical Reviews, *Zentralblatt für Mathematik* and *Referativnyi Żurnal* (*Matematika*), insert annually over 1000 reviews dedicated to integral equations and operators (not counting those papers in which integral equations appear only casually). It is very difficult to sketch adequately the contemporary picture of the research field of integral equations and operators. This should really be the work of a whole team of investigators. I will, however, make an attempt, although convinced that serious shortcomings might occur.

In the early 1960s, a series of research papers due to J. J. Levin and J. A. Nohel pointed out the role of integral equations as tools in the study of the stability of nuclear reactors. Unlike Popov [1], Levin and Nohel based their investigation on the so-called 'energy method'. In other words, a kind of Liapunov functional was used to derive information about the solutions. The research work started by Levin and Nohel at the University of Wisconsin at Madison has been continued by attracting the attention of numerous other researchers. During the 1970s and the 1980s the Madison school concentrated its efforts on problems occuring in continuum mechanics, particularly in viscoelasticity. The recent monograph by Renardy, Hrusa and Nohel [1] provides a picture of the research activities conducted at Madison, though not a complete one.

About the same time, the School of Continuum Mechanics at Carnegie-Melon University in Pittsburgh was directing its efforts towards the foundations of this discipline, making systematic use of the theory of integral equations and the theory of equations with infinite delay. For contributions from this school see, in the list of references, papers under the names of MacCamy, Mizel, Hrusa, and their co-workers (Marcus, Leitman, and others). In relation to this school and contributions to the topic of integral equations or equations with delay we should also add the names of Coleman, Gurtin and Noll.

At Brown University in Providence, Rhode Island, J. K. Hale and many co-workers have made numerous and substantial contributions to the field under discussion. Also, C. M. Dafermos, mostly from the point of view of a researcher in continuum mechanics, has brought valuable contributions to the theory of integral equations (more precisely, integro-partial differential equations).

Another group of researchers interested in the theory of integral equations is the one at Virginia Polytechnic Institute and State University (K. Hannsgen, R. L. Wheeler, T. L. Herdman, M. Renardy and others).

At the Southern Illinois University in Carbondale, a group of researchers are actively participating in the development of the theory of integral and related equations, including investigations in the theory of control systems described by means of integral equations (T. A. Burton, D. Carlson, R. C. Grimmer, C. E. Langenhop).

A considerable number of isolated researchers have conducted valuable work in the field of integral and related equations, in the United States: F. Bloom, F. E. Browder, J. R. Cannon, D. Colton, J. M. Cushing, H. Engler, W. E. Fitzgibbon, H. E. Gollwitzer, C. W. Groetsch, M. L. Heard, A. J. Jerri, G. S. Jordan, R. K. Miller, M. Milman, K. S. Narendra, M. Z. Nashed, A. G. Ramm, W. J. Rugh, I. W. Sandberg, A. Schep, M. Schetzen, V. S. Sunder, and G. F. Webb.

In the Soviet Union, at least four schools of research in integral equations and integral operators have contributed remarkably to the progress of this field during the last three decades: the Krein-Gohberg school, the Krasnoselskii school in Voronež (and then in Moscow), the school in Novosibirsk (Korotkov and his followers) and the school grouped around N. V. Abelev and Z. B. Caljuk (with ramifications in various academic centers of the USSR).

The Krein-Gohberg school had many followers in Odessa, Kishinev, and other centers in the Soviet Union. In the 1970s, I. C. Gohberg emigrated to Israel, where he continues his work on Wiener–Hopf techniques and their generalizations. In particular, the factorization problem has been worked on by Gohberg and his followers from Israel, the United States and the Netherlands. The Kishinev group also continued their research activities, more or less on the same lines.

The school created by M. A. Krasnoselskii in Voronež has also spread to various centers. Among the most remarkable achievements of this group we mention the joint work by Krasnoselskii, Zabreyko, Pustylnik and Sobolevskii [1], as well as basic results obtained by P. P. Zabreyko on nonlinear integral operators. In Moscow, Krasnoselskii and Pokrovskii [1] are conducting research work on 'systems with hysteresis'.

The school in Novosibirsk is concentrating on the theory of Carleman (unbounded) integral operators. This direction is very promising for the near future, when we can expect some applications to the theory of integral equations. So far, only sporadic results have been obtained (see, for instance, Korotkov [2]).

Various interesting results concerning integral and related equations have been obtained by the group led by N. V. Azbelev and Z. B. Caljuk: stability problems, boundary value problems for functional-differential equations, integral representation of solutions, and many other aspects have been emphasized.

In many other academic centers in the USSR, the integral and related equations are cultivated both from a theoretical point of view, and from the point of view of their applications (we are not concerned here with the so-called singular equations which do appear in mechanical problems, and for which a vast literature is available in Russian).

In Italy, where there has always been interest in Volterra equations and their generalizations and applications, a group of scholars in Pisa, Rome and Trento are heavily engaged in developing the theory of abstract Volterra equations (the term abstract should be interpreted here as Banach-space valued), together with their applications to mechanics or population dynamics. Most of the researchers are students of G. DaPrato (M. Iannelli, A. Lunardi, E. Sinestrari, G. DiBlasio, and others). An important reference not belonging to the above category is due to Fichera [1], who has also made other significant contributions to the field.

As mentioned above, other countries participated during the last three decades to the advancement of the theory of integral equations: in Finland (S.-O. Londen, O. J. Staffans, G. Gripenberg) the theory of convolution equations has seen a real development. By means of classical and functional analytic methods, including semigroup theory and Laplace–Fourier transform theory, a large variety of qualitative problems have been successfully investigated. A book by Finnish mathematicians, probably dealing with this kind of investigation, is in preparation. In Japan, special attention has been paid to the theory of equations with infinite delays (J. Kato, T. Naito, Y. Hino, S. Murakami), and functional equations involving integral operators have been investigated by many authors (for instance, N. Hirano, Nobuyuki Kato).

While most contributions to the theory of integral and related equations, published during the last three decades, deal with various applications, it is worth while mentioning the fact that in their attempt to solve such applied problems the investigators made use of quite recent tools created by basic mathematical research (monotone operators, linear and nonlinear semigroups of transformation, as well as many other functional analytic methods). In this way, the theory of integral equations has been considerably developed and enriched.

I will now briefly describe the structure of the book, pointing out directions in which the topics under discussion could be further developed.

The first chapter is introductory, and is aimed at emphasizing the fact that integral equations/operators occur either directly in the description of certain phenomena, or indirectly – when processing other types of functional equations. Also, it contains a rather elementary introduction to the theory of Volterra equations, as well as the Fredholm theory of linear integral equations. The last section of the first chapter deals with Hammerstein equations, basically under the hypotheses adopted by Dolph in his thesis at Princeton (1944). This theory is nonlinear, and it can be dealt with using fairly classical tools. The reader might be surprised by the fact that very little is included in relation to the Hilbert (or Hilbert–Schmidt) theory of symmetric kernels. Indeed, this is one of the most salient parts of the classical theory of linear integral equations, and it should certainly be included in any book on this subject. In order to keep the size of the volume within reasonable dimensions, however, I decided not to include this classical chapter of the theory of integral equations. The interested reader can find numerous sources for this theory, under various basic assumptions on the kernel (besides its symmetry): Cochran [1], C. Corduneanu [6], Fenyo and Stolle [1], Goursat [1], Hamel [1], Hochstadt [1], Hoheisel [1], Kanwal [1], Lalesco [1], Lovitt [1], Mikhlin [1], Petrovskii [1], Schmeidler [1], Smithies [1], Tricomi [1], Vivanti [1].

The second chapter is an auxiliary for the following chapters and contains a series of definitions and properties of some function spaces to be used in this book, some properties of certain integral operators acting on the function spaces introduced earlier, as well as the statements (with indications for the proofs) of several basic results relating to fixed point theorems or monotone operators. This chapter is not intended to be a complete presentation of the topics which are considered, and its only role should be to enable readers to become acquainted with some of the methods and tools to be used in later chapters. On the other hand, special results from functional analysis that are needed only in connection with a single result in the book, have been stated (and reference provided in book form) in subsequent chapters.

The third chapter of the book is entirely dedicated to the (mostly but not only local) theory of Volterra equations, including functional-differential equations that can be reduced to Volterra ones. It turns out that, in the framework of abstract Volterra equations, one can encompass practically all types of equations with delayed argument. I have particularly illustrated the case of equations with infinite (unbounded) delays, but the theory of equations with finite delay is also a special case of abstract functional-differential equations with Volterra (causal) operators. This approach will probably change the way most particular problems are addressed nowadays, introducing a broad unifying idea and providing more generality. There is more to be done in this respect, and not only in connection with the general theory of existence, uniqueness, continuous dependence, etc. Applications of the abstract approach will certainly be extended to the theory of control processes described by this type of equation. The last section of the chapter is devoted to the presentation of an approach based on the singular perturbation method, also in the case of abstract Volterra equations. While this method has been used by several authors in connection with particular classes of Volterra equations, none of the contributions has dealt with the abstract case.

The fourth chapter is a collection of results related to both Volterra and Fredholm equations (particularly in the form $x = f + KNx$, with K a linear integral operator and N a nonlinear Niemytskii operator), which tries to emphasize the connection of these equations to various problems in nonlinear analysis, such as boundary value problems for ordinary differential equations. Admissibility techniques, construction of resolvent kernels with preassigned properties, Hammerstein equations in spaces of measurable functions, asymptotic behavior of solutions for integrodifferential equations on a half-axis, perodic and almost periodic solutions, multivalued integral equations (inclusions) are considered in this chapter. These topics have been recently discussed in the mathematical literature on this subject, and they certainly reflect some current preoccupations in this field of research. Of course, the list of topics could be considerably extended, owing to the significant amount of research work currently being carried out in this field. Let us point out that most of the material included in the fourth chapter is formed on the pattern of finite-dimensional theory, but some topics are suitable for generalization to the infinite-dimensional case.

In the fifth chapter some problems pertaining to the theory of integral equations in a Banach (Hilbert) space are discussed, while some methods like the semigroup theory and monotone operators are illustrated in connection with the theory of integral or related equations. In some cases, as for example in the second section, a general operator approach has been adopted for the Amann's generalization of Hammerstein theory, and then applications are considered to integral equations. The semigroup method is presented in connection with the problem of existence of the resolvent kernel (which is still far from a satisfactory solution in the infinite-dimensional case). The nonlinear semigroups appear naturally in discussing the dynamics described by a nonlinear time-invariant integral equation of Volterra type. Integrodifferential equations in Hilbert space are also discussed in this chapter, with applications to the existence theory for integro-partial differential equations (as those occurring in continuum mechanics). Much more remains to be done in the case of integral and related equations in Banach/Hilbert spaces, since the existing results belong either to classes of rather special equations, or relate to equations with bounded operators for which applications appear very seldom.

The last chapter of the book is devoted to some applications that the integral