RIEMANNIAN GEOMETRY
AND THE
TENSOR CALCULUS

An Introduction to

RIEMANNIAN GEOMETRY

AND THE

TENSOR CALCULUS

by

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To

DEAN L. P. EISENHART

and

PROFESSOR O. VEBLEN

WHOSE WORK WAS
THE INSPIRATION TO WHICH THE
WRITING OF THIS BOOK
WAS LARGELY DUE

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PREFACE

My object in writing the following pages has been to provide a book which will bridge the gap between differential geometry of Euclidean space of three dimensions and the more advanced work on differential geometry of generalised space. The subject is treated with the aid of the Tensor Calculus, which is associated with the names of Ricci and Levi-Civita; and the book provides an introduction both to this calculus and to Riemannian geometry. I have endeavoured to keep the analysis as simple as possible, and to emphasise the geometrical aspect of the subject. The geometry of subspaces has been considerably simplified by use of the generalised covariant differentiation introduced by Mayer in 1930, and successfully applied by other mathematicians. In the main I have adopted the notation and methods of the Italian and Princeton schools; and I have followed the example of Levi-Civita in using a Clarendon symbol to denote a vector, which has both covariant and contravariant components.

For the greater part of a century multidimensional differential geometry has been studied for its own intrinsic interest; and its importance has been emphasised in recent years by its application to general theories of Relativity. I hope, therefore, that this volume will be of service also to students who propose to devote their attention to the mathematical aspect of Relativity. A historical note has been written in order to add to the interest of the book. This is placed at the end, rather than at the beginning, as some knowledge of the subject is necessary for its appreciation.

C. E. W.

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