Chapter I

SETS*

§ 1. THE CALCULUS OF SETS

1. The object of the calculus described in § 1 is a practical one—to shew how complicated properties of sets may be deduced by formal rules from a small number of properties which are sufficiently simple to be self-evident. The propositions accepted as self-evident (lettered A to F) are not intended as a system of axioms—a much smaller number would suffice for that purpose—but as a convenient body of "standard forms" for use in symbolical work.

2. A set (or class or aggregate) is to be thought of not as a heap of things specified by enumerating its members one after another, but as something determined by a *property*, which can be used to test the claim of any object to be a member of the set. Thus the set of even integers is determined by the property of being twice some other integer, the algebraic numbers by the property of satisfying a polynomial equation with integral coefficients. Two properties determine the same set if they are "formally equivalent", i.e. if no object has one property without having the other.

The symbol $x \in A$ means "x is a member (or element) of A". The set which has only the single member a is denoted by (a). Thus $x \in (a)$ means simply $x = a$.

3. The symbol $A \subseteq B$ (or $B \supseteq A$) means that, for every x,

$$x \in A \quad \text{implies} \quad x \in B.$$

It is usually read "A is contained in B", or "A is a subset of B", but, as the form of the symbol suggests, identity is not excluded.

* The subjects dealt with in this preliminary chapter are (1) the calculus or algebra of sets, and (2) the distinction between enumerable sets and others. Readers who are familiar with these matters should omit this chapter, but should note the definitions here adopted for the symbols $A \subseteq B$ and $A \subset B$, pp. 1, 2; $B - A$, p. 5; and $\mathscr{C}A$, p. 5.

The symbol $A \subset B$ (rarely used in this book) means "$A \subseteq B$ but $A \neq B$", and is read "A is a proper subset of B".[(1)]*

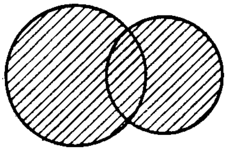
A 1. *$A \subseteq A$.*

2. *If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

3. *$A = B$ if, and only if, $A \subseteq B$ and $B \subseteq A$.*

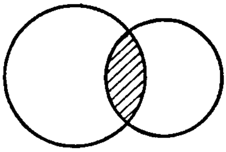
These properties of sets must be accepted as self-evident. (A 3 may, if preferred, be regarded as a definition of equality between sets.)

4. The *join* or *union*, $A \cup B$, of the sets A and B is the set of all members of either set. Thus "$x \in A \cup B$" means "$x \in A$ or



$A \cup B$

Fig. 1



$A \cap B$

Fig. 2

$x \in B$ (or both)". For example, if A is the set of real numbers between 0 and 2, and B the set between 1 and 3, $A \cup B$ is the set between 0 and 3.

The *common part*, or intersection, $A \cap B$, of the sets A and B is the set of things belonging to both A and B, i.e. "$x \in A \cap B$" means "$x \in A$ and $x \in B$".

The principal identical relations involving $\cup$ and $\cap$ are

B 1. $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$

2. $A \cup B = B \cup A, \quad A \cap B = B \cap A.$

3·1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

3·2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$

4. $A \cup A = A, \quad A \cap A = A.$

It will be noticed that $\cup$ and $\cap$ behave in many ways like ordinary addition and multiplication, B 1, 2 and 3·1 correspond-

* The numbers (1), (2), etc., refer to the notes at the end of the book.

ing to the associative, commutative and distributive laws. In accordance with this analogy, AB will be used as an abbreviation for $A \cap B$, with the bracketing convention $AB \cup C = (AB) \cup C$. Properties B 1–4 may then be summed up in the rules that (*a*) expressions may be transformed as in ordinary algebra, except that there is no cancelling, i.e. neither of the relations $AB = AC$ and $A \cup B = A \cup C$ implies $B = C$; and (*b*) "multiples" and "powers" of A are to be replaced by A (B 4). The dual distributive law B 3·2 is deducible from the rest (see Example 1).

The following inclusion relations hold:

B 5. $A \subseteq A \cup B, \quad A \supseteq A \cap B.$

6·1. If $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.

6·2. If $A \supseteq C$ and $B \supseteq C$ then $A \cap B \supseteq C$.

All the formulae B may be accepted as self-evident as they stand, or referred back to propositions of logic. For example, if the "definitions" of $A \subseteq B$ and of $x \in A \cup B$ are inserted in B 6·1, it becomes "if $x \in A$ implies $x \in C$, and $x \in B$ implies $x \in C$, then ($x \in A$ or $x \in B$) implies $x \in C$".

If A is a finite set, with members $a, b, \ldots, k$, then

$$A = (a) \cup (b) \cup \ldots \cup (k),$$

which will be abbreviated to $\{a, b, \ldots, k\}$.

Examples. 1. Prove B 3·2 from A and the rest of B 1–6.

$$(A \cup B)(A \cup C) = A \cup AB \cup AC \cup BC, \quad \text{by B 1–4, without 3·2.}$$

Since $AB \subseteq A$, $A \cup AB = A$ by B 6·1, and hence

$$(A \cup AB) \cup AC = A \cup AC = A,$$

giving the result.

2. A necessary and sufficient condition that $A \subseteq B$ is that $A \cup B = B$.

If $A \cup B = B$, $A \subseteq B$ by B 5. If $A \subseteq B$, $A \cup B \subseteq B$ by A 1 and B 6·1, and $B \subseteq A \cup B$ by B 5.

Exercises. 1. Prove B 4 formally from A and the rest of B 1–6 (excluding B 3·2).

2. A necessary and sufficient condition that $A \subseteq B$ is that $AB = A$.

5. The *null-set*, or *empty set*, denoted by 0, has no members and is contained in every set:

C. $0 \subseteq A$: *the null-set is a subset of every set.*

When sets are regarded as collections or heaps of things a set with no members is a rather shadowy or even paradoxical entity, but its mysterious quality disappears if statements about sets are interpreted as statements about properties. Let p be called a *null-property* if it is not possessed by any object. Examples are: being greater than 3 and less than 2, or being a zero of e^z. Such properties are frequently considered in mathematics, particularly in proofs by *reductio ad absurdum*. Any two null-properties are "formally equivalent", in the sense of para. 2, and therefore all these properties determine the same set, which is called the null-set.

To arrive from this definition at proposition C it is necessary to consider more closely the interpretation of the symbol $A \subseteq B$. The meaning assigned to it was: for every x, $x \in A$ implies $x \in B$. This means that $x \in B$ unless "$x \in A$" is false, i.e. "$x \in B$" *is true, or* "$x \in A$" *is false*. This final form may be taken as the basic meaning of $A \subseteq B$, and from it it is clear that $0 \subseteq A$. For since, for all x, "$x \in 0$" *is* false, the proposition

$$\text{"}x \in A\text{" is true or "}x \in 0\text{" is false}$$

is true, whatever the set A may be.

From C and B 6·1 and 6·2 it follows that

$$A \cup 0 = A, \quad A \cap 0 = 0.$$

Thus the formal properties of the null-set justify the symbol 0.

Two sets are said to *meet* (or intersect) if they have at least one common member. It follows from the definition of the null-set that the necessary and sufficient condition that A and B meet is that $AB \neq 0$. Two sets that do not meet are said to be *disjoint*.

Note. In work that is not purely symbolical the symbol $A \subseteq B$ is often replaced by the words "all a's are b's"—for example, "the set of all parabolas is contained in the set of all conics" by "all parabolas are conics". It must, however, be borne in mind that if A is the null-set "all a's are b's" is to be regarded as true whatever B may be. Thus all zeros of e^z are real and positive, because e^z has no zeros. All zeros of e^z are also real and negative, and there is no contradiction between the statements, because it is not asserted that any actual number is both positive and negative, but only that the set of zeros of e^z (i.e. the null-set) is a subset of both the other sets of numbers.

6. If $A \subseteq S$ the set of elements of S not belonging to A is called the *complement*, or *residual set*, of A with respect to S. It is denoted by $S-A$; but if S is supposed fixed, $S-A$ may also be denoted by $\mathscr{C}A$.

Besides the obvious properties

D 1. $\mathscr{C}S = 0, \quad \mathscr{C}0 = S,$

2. $A \cup \mathscr{C}A = S, \quad A \cap \mathscr{C}A = 0,$

3. $\mathscr{C}(\mathscr{C}A) = A,$

4. *If* $A \subseteq B$ *then* $\mathscr{C}B \subseteq \mathscr{C}A,$

the complement has the important property of interchanging $\cup$ and $\cap$:

D 5. $\mathscr{C}(A \cup B) = \mathscr{C}A \cap \mathscr{C}B, \quad \mathscr{C}(A \cap B) = \mathscr{C}A \cup \mathscr{C}B.$

(See Figs. 1 and 2, where S may be taken to be the whole plane.) This proposition corresponds to the theorem in logic that "not (p or q)" is equivalent to "not p and not q", and "not (p and q)" to "not p or not q".

* Theorem 6·1. *If* $A \cup X = S$, *and* $AX = 0$, *then* $X = \mathscr{C}A$.

By the first of the given equations, $X \subseteq S$. Multiplying the first equation of D 2 by X and using $AX = 0$, we obtain

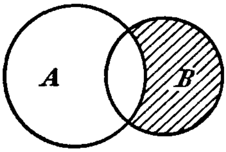
$$X \cap \mathscr{C}A = X.$$

Multiplying the first of the given equations by $\mathscr{C}A$ and using $A \cap \mathscr{C}A = 0$, we get $X \cap \mathscr{C}A = \mathscr{C}A$. Hence $X = \mathscr{C}A$.

If A and B are any sets, $B - A$ denotes the set of elements of B not belonging to A (Fig. 3). Thus if complements are formed with respect to a set containing A and B as subsets, $B - A = B\mathscr{C}A$.

* I.e. Theorem 1 of para. 6 (of Ch. I), referred to in this chapter as 6·1, in others as I. 6·1.

Evidently $A-A=0$ and $A-0=A$. However, this "minus" sign does not combine with $\cup$ in the way that the "plus"-analogy might suggest. For example, $(A\cup A)-A=0$, but $A\cup(A-A)=A$. For this reason it is best to begin formal proofs by replacing differences $B-A$ by the equivalent $B\mathscr{C}A$.



B-A
Fig. 3

Examples. (All complements are formed with respect to an arbitrary set, S, containing A, B and C, except in Example 4.)

1. $A(B-C)=B(A-C)=AB-C$, for all three sets are $AB\cap\mathscr{C}C$.

2. $B-A=B-AB$.

$$\begin{aligned}B-AB &= B\cap\mathscr{C}(AB)\\ &= (B\cap\mathscr{C}A)\cup(B\cap\mathscr{C}B)\\ &= B\cap\mathscr{C}A = B-A.\end{aligned}$$

3. A necessary and sufficient condition that $AB=0$ is that $A\subseteq\mathscr{C}B$.

If it is given that $A\subseteq\mathscr{C}B$, multiply both sides by B. If it is given that $AB=0$, multiply both sides of $A\subseteq B\cup\mathscr{C}B$ by A.

4. If $C\subseteq B\subseteq A$ then $(A-B)\cup(B-C)=A-C$. If complementation is with respect to A, the left-hand side is $\mathscr{C}B\cup B\mathscr{C}C$. Since $\mathscr{C}B\subseteq\mathscr{C}C$, this is
$$(\mathscr{C}B\cap\mathscr{C}C)\cup B\mathscr{C}C=\mathscr{C}C=A-C.$$

5. *The propositions* D 1 *and* D 3–5 *can be deduced formally from* A, B *and* D 2. Since $A\mathscr{C}A\subseteq A$ (by B 5), C follows immediately from D 2. The theorem 6·1 is proved as in the text. It uses only A–C and D 2.

Proof of D 1. Since $S\cup 0=S$ and $S0=0$, it follows from 6·1 that $\mathscr{C}S=0$ and $\mathscr{C}0=S$.

Proof of D 3. By D 2 the equations
$$\mathscr{C}A\cup X=S,\quad \mathscr{C}A\cap X=0$$
are satisfied by $X=A$. Therefore by 6·1, $A=\mathscr{C}(\mathscr{C}A)$.

Proof of D 4. Since $A\subseteq B$, $S=A\cup\mathscr{C}A\subseteq B\cup\mathscr{C}A$. Multiply both sides by $\mathscr{C}B$:
$$\mathscr{C}B\subseteq\mathscr{C}B\cap\mathscr{C}A\subseteq\mathscr{C}A.$$

Proof of D 5. Let $A'=\mathscr{C}A$ and $B'=\mathscr{C}B$. Then
$$(A\cup B)\cap(A'B')=(AA'\cap B')\cup(BB'\cap A')=0,$$
$$(A\cup B)\cup(A'B')\supseteq(AB'\cup A'B\cup AB)\cup A'B',$$

since all the sets in the bracket on the right are contained in A or B,

$$= (A \cup A') \cap (B \cup B') = S.$$

Since A, B and $A'B'$ are all contained in S, it follows that

$$(A \cup B) \cup A'B' = S.$$

Hence by 6·1 $A \cup B$ and $A'B'$ are complementary sets. This gives the first part of D 5 immediately, and the second part on interchanging dashed and undashed letters.

Exercises. 1. $A(B-C) = AB-AC$.

2. $\mathscr{C}(A-B) = B \cup \mathscr{C}A$.

3. $(A-B) \cup (A-C) = A-BC$.

4. $(A-C) \cup (B-C) = (A \cup B)-C$.

5. $(A-B) \cup (B-A) = (A \cup B)-AB$.

6. $A-(A-B) = AB$.

7. $\mathscr{C}(A_1 \cup A_2 \cup \ldots \cup A_k) = \mathscr{C}A_1 \mathscr{C}A_2 \ldots \mathscr{C}A_k$.

8. $A \cup (B-A) = A \cup B$, $A(B-A) = 0$. Prove that the whole of D 1–5, and the equation $B-A = B\mathscr{C}A$ can be deduced formally from these two relations together with A, B and the definition "$\mathscr{C}A = S-A$ if $A \subseteq S$". [First prove that the equations

$$A \cup X = A \cup B, \quad AX = 0$$

have at most one solution.]

7. *Duality.* The calculus that has now been developed (Boolean Algebra[2]) has a duality property which has probably already been observed by the reader. If in any theorem of the Algebra all differences are expressed in terms of complements with respect to a fixed set S, and then the symbols

$$\left.\begin{array}{l} \cup \text{ and } \cap \\ 0 \text{ and } S \\ \subseteq \text{ and } \supseteq \end{array}\right\} \text{ are everywhere interchanged,}$$

the result is also a true theorem of the Algebra.

Since no appeal is made to the duality property in this book a general proof (which would require a more exact definition of Boolean Algebra) is not given. (Cf. Note 2.)

8. It is frequently necessary to consider sets whose members are themselves sets of things. If **M** is such a set of sets, the members of **M** (i.e. the sets of "things") are usually denoted by a suffix notation, A_x. The suffix x may range through any set B, e.g. the integers from 1 to k, all the positive integers, all the real numbers, etc. When this notation is used for the members of **M** the set **M** itself is denoted by $\{A_x\}$.

The *union*
$$\bigcup_{x \in B} A_x$$
is the set of all members of the sets A_x; i.e.
$$\text{“}z \in \bigcup_{x \in B} A_x\text{”}$$
means "for some x of B, $z \in A_x$". The *intersection*
$$\bigcap_{x \in B} A_x$$
is the set of elements that belong to all the A_x; i.e.
$$\text{“}z \in \bigcap_{x \in B} A_x\text{”}$$
means "for every x of B, $z \in A_x$". The notations for union and intersection may be abbreviated to $\bigcup_x A_x$ and $\bigcap_x A_x$, $\bigcup_x A_x$ and $\bigcap_x A_x$, or even $\bigcup A$ and $\bigcap A$, when the meaning is clear. When the suffixes are positive integers the union is denoted by
$$\bigcup_1^k A_n \quad \text{or} \quad \bigcup_1^\infty A_n,$$
and the intersection similarly; but it is to be emphasised that the infinite union and intersection are not derived from the finite ones by any limiting process, but have an independent definition of their own.

The definitions evidently agree with those previously given for "union" and "intersection" when the number of sets is finite.

Example. If A_n is the set of roots of the equation $z^n = 1$, $\bigcup_1^\infty A_n$ is the set of numbers $e^{2\pi i \alpha}$, where α takes all rational values; and $\bigcap_1^\infty A_n$ is the single number 1.

The formal properties of $\bigcup$ and $\bigcap$ are:

E 1. *If* $a \in B$, $\bigcap_{x \in B} A_x \subseteq A_a \subseteq \bigcup_{x \in B} A_x$.

2·1. *If, for every a of B,* $A_a \subseteq C$, *then* $\bigcup A_x \subseteq C$.

2·2. *If, for every a of B,* $A_a \supseteq C$, *then* $\bigcap A_x \supseteq C$.

These propositions may be "translated" in the usual way; e.g. E 2·1 states that if $a \in B$ implies $A_a \subseteq C$, then "$z \in A_a$ and $a \in B$" implies $z \in C$.

F 1. *If* $A_x \subseteq B_x$ *for each* x, $\bigcup A_x \subseteq \bigcup B_x$ *and* $\bigcap A_x \subseteq \bigcap B_x$.

2. $\bigcup(A_x \cup B_x) = \bigcup A_x \cup \bigcup B_x$.

3·1. $\bigcap(A \cup B_x) = A \cup \bigcap B_x$.

3·2. $\bigcup(AB_x) = A \bigcup B_x$.

4. *If S contains all the sets* A_x, *then* $\bigcup A_x$ *and* $\bigcap(S - A_x)$ *are complementary sets in* S; *i.e. if* $\mathscr{C}$ *denotes the complement in* S, $\mathscr{C}(\bigcup A_x) = \bigcap(\mathscr{C} A_x)$.

As a final example of the use of the "calculus" it will now be shewn that the propositions F are formally derivable from A–E. From this and other examples that have been given in this section it follows that all the "standard forms" A–F can be derived formally from A, B (without B 4), D 2 and E.

Proof of F 1. For every a, $A_a \subseteq B_a \subseteq \bigcup B_x$ and $\bigcap A_x \subseteq A_a \subseteq B_a$: apply E 2.

Proof of F 2. Since $A_a \subseteq \bigcup A_x$ and $B_a \subseteq \bigcup B_x$, $A_a \cup B_a \subseteq \bigcup A_x \cup \bigcup B_x$. Therefore by E 2·1

$$\bigcup(A_x \cup B_x) \subseteq \bigcup A_x \cup \bigcup B_x.$$

The other half follows from F 1.

Proof of F 3·1. Let $X = \bigcap(A \cup B_x)$. Then, by F 1, $A \subseteq X$ and $\bigcap B_x \subseteq X$, and therefore

$$A \cup \bigcap B_x \subseteq X.$$

If B_a is one of the sets B_x, $X \subseteq A \cup B_a$, and therefore

$$X - A \subseteq (A \cup B_a) - A \subseteq B_a.$$

Hence $X - A \subseteq \bigcap B_x$, and therefore $X \subseteq A \cup \bigcap B_x$.

Proof of F3·2. If $Y = \bigcup(AB_x)$ then, by F1, $Y \subseteq \bigcup B_x$ and $Y \subseteq A$, and therefore $Y \subseteq A \bigcup B_x$. If S is a set containing all the sets involved (e.g. $S = A \cup \bigcup B_x$),

$$B_a = AB_a \cup (S-A)\, B_a$$
$$\subseteq Y \cup (S-A).$$

Therefore $$\bigcup B_x \subseteq Y \cup (S-A),$$

and $$A \bigcup B_x \subseteq AY \subseteq Y.$$

Proof of F4. Let $\bigcup A_x = X, \bigcap(S-A_x) = Y$. Then clearly $X \cup Y \subseteq S$, and

$$X \cup Y = X \cup \bigcap(S-A_x) = \bigcap(X \cup (S-A_x))$$
$$\supseteq \bigcap(A_x \cup (S-A_x)) = S;$$
$$XY = Y \bigcup A_x = \bigcup(YA_x) \subseteq \bigcup((S-A_x) \cap A_x) = 0.$$

The result now follows by 6·1.

§ 2. ENUMERABLE AND NON-ENUMERABLE SETS

9. A (1, 1)-*transformation*, f, of a set A on to a set B (or a (1, 1)-*correspondence* between the sets) is determined if with every element x of A there is associated an element $f(x)$ of B, called the

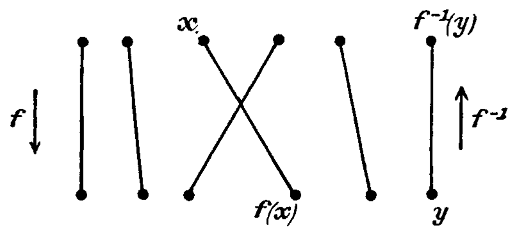


Fig. 4. (1, 1)-correspondence

image of x, in such a way that each element y of B is the image of just one element of A (which is called $f^{-1}(y)$). The condition is symmetrical between A and B, and f^{-1} is a (1, 1)-transformation of B on to A.

The sets A and B are *similar*, ($A \operatorname{sm} B$), if a (1, 1)-correspondence can be set up between them. Clearly if $A \operatorname{sm} B$, then $B \operatorname{sm} A$. If $A \operatorname{sm} B$ and $B \operatorname{sm} C$, then $A \operatorname{sm} C$; for if f and g are (1, 1)-transformations of A on to B and B on to C respectively, gf is a (1, 1)-transformation of A on to C.

Examples. A (1, 1)-correspondence is set up between the set of all positive integers, I, and the set of positive even integers, E, by map-