CHAPTER 1

INTRODUCTION

1. Parallelism in the plane

When the study of plane Euclidean geometry is being developed by coordinate methods, there are two normally accepted procedures for dealing with the phenomenon of parallelism. In the first of these procedures, we regard the Euclidean plane, effectively, as a real projective plane with one line removed—a line which we denote by l_∞; we can then regard the projective plane, conversely, as an *extended Euclidean plane*; and thereafter we define parallel lines as lines whose closures in the extended Euclidean plane meet on l_∞. By the second procedure we first introduce a distance function and the concept of perpendicularity, showing that the shortest distance from a point to a line is the perpendicular distance; and we then say that two lines are parallel if each point of either is at the same perpendicular distance from the other.

When we come to study non-Euclidean geometry, it is natural to investigate the possibility of generalizing one or other of the above approaches to parallelism. A non-Euclidean plane, as we may recall, is basically a real projective plane in which there is singled out a particular polarity ω—the *absolute polarity*; that is to say, in a chosen system of homogeneous coordinates (x_0, x_1, x_2) in the plane, there is singled out a bilinear relation

$$\sum_{i=0}^{2}\sum_{j=0}^{2} a_{ij}x_i x'_j = 0 \qquad (1.1)$$

such that the matrix $A = (a_{ij})$ is real, non-singular and symmetric; and two points (x_0, x_1, x_2) and (x'_0, x'_1, x'_2) are said to be *conjugate* with respect to the polarity ω if their coordinates satisfy (1.1). The points of the plane that are conjugate to themselves are those of the (non-singular) *absolute conic* Ω whose equation is

$$\sum_{i=0}^{2}\sum_{j=0}^{2} a_{ij}x_i x_j = 0;$$

and the non-Euclidean plane is said to be *hyperbolic* or *elliptic* according as Ω does or does not contain any real points; or, as we usually say, according as Ω is a *real* or a *virtual*† conic.

In the hyperbolic case Ω disconnects the projective plane into two regions, one homeomorphic with a disk and the other with a Möbius strip; but it is customary to think of the hyperbolic plane as properly consisting only of the points of the former region, those of the latter region being called *ultra-infinite points* and those of Ω *points at infinity*. The elliptic plane, on the other hand, consists of *all* the points of the underlying projective plane.

The *non-Euclidean distance* $d(P,Q)$ between points P and Q of a non-Euclidean plane is defined in terms of the cross-ratio $\{P,Q;Q',P'\}$, where P' and Q' are the points of the line PQ that are conjugate to P and Q respectively in the polarity ω. The usual definition is

$$\left.\begin{matrix}\cosh^2(d(P,Q))\\ \cos^2(d(P,Q))\end{matrix}\right\} = \{P,Q;Q',P'\},$$

the hyperbolic or trigonometric cosine being taken according as the plane is hyperbolic or elliptic.‡ The use of the term 'point at infinity' for a point of Ω (in the hyperbolic case) is justified by the fact that $d(P,Q)$ tends to infinity if P is kept fixed while Q is made to approach a point of Ω.

After these preliminaries we may now consider possible generalizations to non-Euclidean geometry of the two ways of introducing parallel lines in the Euclidean plane. By analogy with the first of these ways we may define two lines of a non-Euclidean plane to be parallel if they meet 'at infinity', i.e. on Ω. In the hyperbolic case this definition makes sense, and its consequences have been fully worked out in treatises on non-Euclidean geometry (cf., for example, Coxeter [**6**] which contains an extensive bibliography). In the elliptic case, however, the definition is empty because Ω contains no real points.

† Since virtual conics contain no real points, they do not exist (as loci) in the same sense as real conics. They are associated, however, with well-defined real polarities and thereby acquire a conventional 'existence' as conics in the real projective plane.

‡ The formulae given above do not define $d(P, Q)$ unambiguously in either case. For the conventions necessary to their interpretation we refer the reader to the account given in Coxeter [**6**].

As regards the possible generalization of the 'equidistance' definition of parallel lines, we note first that *perpendicular lines* in either kind of non-Euclidean geometry are defined to be lines which are conjugate with respect to the absolute polarity ω, and that the shortest distance from a point to a line is the perpendicular distance. We then find, however, that no two distinct lines can ever have the 'equidistance' property; and, in fact, the locus of a point which is at a fixed perpendicular distance from a given line – *an equidistant curve* – is (algebraically) a conic touching Ω at its intersections with the line. There is, accordingly, no analogue, in either kind of non-Euclidean plane, of the equidistance definition of parallelism in the Euclidean plane.

2. Parallel lines in 3-dimensional space

Consider now the corresponding problems in 3-dimensional space. Parallel lines in Euclidean 3-dimensional space can be introduced in either of two ways, either as lines that meet on a conveniently introduced 'plane at infinity' or as lines that are everywhere equidistant; and we look again at the possibility of adapting one or other of these definitions to the circumstances of 3-dimensional non-Euclidean geometry.

A 3-dimensional non-Euclidean space is a 3-dimensional real projective space in which there is singled out a certain fixed (non-singular) quadric polarity ω—the *absolute polarity*—given in a chosen coordinate system by an equation

$$\sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_i x'_j = 0,$$

where $A = (a_{ij})$ is a non-singular real symmetric matrix. We again distinguish different kinds of non-Euclidean space according to the nature of the non-singular *absolute quadric* Ω whose equation is

$$\sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_i x_j = 0.$$

Specifically, the space is *elliptic* if Ω is a virtual quadric (containing no real points) and *hyperbolic* if Ω is a real quadric of the kind that is homeomorphic with an ordinary 2-dimensional sphere; and in the latter case we again distinguish (as in the plane) between ordinary, infinite and ultra-infinite points.† *Non-Euclidean distance* is defined, in either kind of space, by the same formulae as in the plane; and two lines p and q are defined to be *perpendicular* if they are conjugate with respect to ω, i.e. if each meets the polar line of the other. It is easy to verify, then, that the shortest distance from a point to a line is the perpendicular distance.

As regards the possible definitions of parallel lines, the first possibility is to define two lines as being parallel if they meet on Ω; and, just as in the plane, this definition makes sense in the hyperbolic but not in the elliptic case. With the equidistance definition, however, an exciting new possibility arises, of which there was no analogue in the plane. The locus ϕ of points at a fixed perpendicular distance from a given line l – an *equidistant surface* – turns out to be a quadric surface which (algebraically) touches Ω at its two intersections with l; and there is now the possibility that this quadric ϕ may be of the type that contains real generating lines. It then turns out that this possibility is realized if (and only if) the non-Euclidean space is elliptic. To recapitulate formally:

In 3-dimensional elliptic space, the locus of points at a fixed perpendicular distance from a given line l is a real quadric surface ϕ with (*two systems of*) *real generating lines.*

Any line of either generating system on ϕ is therefore parallel to l in the 'equidistance' sense, and it is said to be *Clifford parallel*‡ to l. Plainly, through any general point P (of elliptic space) there pass two lines that are Clifford parallel to a given line l, one generator of each system on the quadric such as ϕ that contains P; and it is possible to introduce conventions whereby one of these lines can be said to be *left-parallel* to l and the other *right-parallel* to l. Left-parallelism and right-parallelism each turn out to be

† There is, of course, another kind of real non-singular quadric in real projective space, namely the kind – homeomorphic with a torus – which possesses two systems of real generating lines; but for various reasons that need not detain us here, this kind of quadric does not give rise to an acceptable kind of non-Euclidean geometry.

‡ In the older literature, Clifford parallel lines were sometimes called *paratactics*.

transitive relations, though Clifford parallelism without qualification is not. An important consequence of the above remarks is that there exist *fibrations* of elliptic 3-space by systems of mutually left- (or right-) parallel lines, where by a fibration we mean a system of which exactly one line passes through each point of space (with no exceptions).

3. Clifford's original definition of parallelism

We now refer to another way of approaching the concept of Clifford parallelism, more in the spirit of Clifford's original discovery (cf. Clifford [**5**]). Suppose that we are given a point P and a line q in elliptic 3-space, and let q' be the polar line of q in the absolute polarity ω. Then the transversal line from P to q and q' is the perpendicular from P to q, and the shortest distance from P to q is measured along this perpendicular. Now suppose that P moves along a line p whose polar line is p'. It can be shown then that the distance from P to q varies continuously as P moves along p, and it attains its maximum and minimum values when P lies on one of the two transversal lines t and t' that can be drawn to meet p, p', q and q'. These lines t and t' are the common perpendicular transversals of p and q; and, provided p and q are in sufficiently general position, there are no others. But, as Clifford noticed, there is a case of exception in which p, p', q and q' belong to the same regulus of generators of a ruled quadric R. In this case p and q possess an infinite number of common perpendicular transversals – the lines of the regulus on R complementary to that containing p, q, p' and q' – and the distances between p and q along these transversals are all equal. These observations lead to the following projective characterization of Clifford parallel lines:

Clifford's definition of parallelism: *Two lines p and q are* (*Clifford*) *parallel if* (*and only if*) *they and their polar lines with respect to ω are four generators of one regulus on a ruled quadric R.*

A quadric such as R, i.e. one which contains two Clifford parallel lines and their polars, possesses the following two (characteristic) properties:

(*a*) Any two non-intersecting generators of R are Clifford parallel; more precisely, the generators of one regulus on R are all mutually left-parallel, while those of the other are mutually right-parallel; and

(*b*) R is autopolar with respect to ω, i.e. it contains the polar line of every one of its generators.

A quadric with properties (*a*) and (*b*) will be called a *Clifford quadric*,† and each of its two reguli will be called a *Clifford regulus*. In the higher dimensional analogues of Clifford parallelism which we shall be studying, it will be found that the appropriate analogue of a Clifford regulus plays a fundamental role.

4. Isoclinic planes in Euclidean space E_4

Further insight into Clifford parallelism may be gained by considering the problem, in 4-dimensional Euclidean space E_4, of defining the angle between two planes (2-dimensional subspaces of E_4).

If π_1 and π_2 are two non-orthogonal planes through a point O of E_4, then one may consider a variable vector v through O in π_1 and calculate the angle between v and its orthogonal projection on π_2. This angle, as it turns out, varies with v in general, achieving its maximum and minimum values when v lies in one or other of two mutually perpendicular directions in π_1. Thus from this approach it is not clear how to define the angle between π_1 and π_2. (For the answers to this and related questions the reader may consult Forsyth [**10**].) There is a case of exception, however, that can and does occur, in which the angle θ between v and its projection on π_2 is independent of the choice of v in π_1. Here there is no doubt as to how to define the angle between π_1 and π_2; and we say that π_1 and π_2 are *isoclinic* at an angle θ.

It is not our purpose to investigate the properties of isoclinic planes in any detail; this has been done elsewhere (see, for example, Forsyth [**10**] and Manning [**19**]). We only wish to point out that Euclidean space E_4 may be thought of as acquiring its Euclidean structure in terms of a 'solid at infinity' together with

† We should point out that Coxeter [**6**] uses the term 'Clifford surface' to describe any quadric which possesses property (*a*) but not necessarily (*b*).

a virtual quadric Ω in this solid; and the solid at infinity has then the structure of an elliptic 3-dimensional space. It may now be verified easily that *two planes of E_4 through O are isoclinic if and only if they meet the elliptic 3-dimensional space at infinity in Clifford parallel lines.*

To mention only one consequence of this, consider a fibration of the solid at infinity by a system of mutually left-parallel lines, say; and join these by planes to a fixed origin O in E_4. Exactly one of these planes will pass through any point of E_4 other than O. Thus if we consider a 3-dimensional sphere S^3 in E_4 with centre O, we see that the planes in question will cut S^3 in a family of non-intersecting great circles such that exactly one passes through any point of S^3. The decomposition space of this family of circles turns out to be a 2-dimensional sphere S^2; and what we have, in effect, is the well-known Hopf fibering of S^3 by 1-spheres over S^2. In terms of this construction one can further establish the existence of three mutually orthogonal fields of unit tangent vectors on S_3, thus demonstrating its parallelizability.

5. Generalization of Clifford parallelism

A $(2n-1)$-dimensional elliptic space EL_{2n-1} is a $(2n-1)$-dimensional real projective space in which a certain virtual quadric is assigned the role of absolute quadric Ω and the associated real non-singular polarity is the absolute polarity ω. An $(n-1)$-dimensional subspace Π of EL_{2n-1} has a polar space Π' with respect to ω, also of dimension $n-1$, and a line is said to be perpendicular to Π if it meets Π'. If we define distance in the same way as for the elliptic plane, we find that the shortest distance to Π from a point is the perpendicular distance, and we may make the following definition:

Two $(n-1)$-dimensional subspaces Π and Π_1 of EL_{2n-1} are to be called *Clifford parallel* if the perpendicular distance to Π_1 from any point P of Π is independent of the choice of P in Π.

This is no more than a straightforward generalization of the original concept of Clifford parallelism, and it is therefore remarkable that it was not systematically studied and developed until recently. The recent development, due to Wong [**27**], is coupled with a parallel study of isoclinic n-dimensional subspaces of Euclidean space E_{2n}, the connection with Clifford parallelism being analogous to that which we have described above (§ 4) for $n = 2$. Wong's development is primarily directed to the discovery of systems – more particularly maximal systems – of mutually Clifford parallel $[n-1]$'s† in EL_{2n-1}; and we may summarize his findings briefly as follows:

(i) For all values of n (> 1) there exist 1-dimensional systems of mutually Clifford parallel $[n-1]$'s in EL_{2n-1}. Important among such systems are those which Wong calls *additive sets*, and these have properties very similar to those of Clifford reguli in the case $n = 2$.

(ii) For given n there exists an r-dimensional system of mutually Clifford parallel $[n-1]$'s in EL_{2n-1} if and only if there exists a set of $r-1$ skew-symmetric real orthogonal matrices of order n which anticommute by pairs. The algebraic problem so arising is that of solving the well-known Hurwitz–Radon matrix equations, and the pairs of values (n, r) for which a solution exists are known‡ (cf. the Appendix to this book).

(iii) Except when $n = 2, 4$ or 8, there exist no n-dimensional systems of mutually Clifford parallel $[n-1]$'s in EL_{2n-1}, and consequently no fibrations of EL_{2n-1} based on such systems; but when $n = 2$, 4 or 8, such fibrations (or *space-filling systems*) do exist. The fibrations for $n = 4$ and $n = 8$ correspond, by a construction similar to that described above for $n = 2$, to the Hopf fiberings of the 7-sphere S^7 by S^3's over S^4 and of the 15-sphere S^{15} by S^7's over S^8.

Since Wong's treatment of Clifford parallelism, and the formulation of his results, were predominantly algebraic and limited to geometry over the real field, it seemed to the present authors that the subject invited an alternative development

† The symbol $[r]$ will be used in this book to denote an r-dimensional (flat) space.

‡ When $n = 3$, the maximum possible value of r is 1; so that in this, the next simplest case of Clifford parallelism after the classical case, the phenomenon is relatively uninteresting. This may partly explain why the problem of generalizing Clifford parallelism did not attract attention for such a long time.

within the general framework of complex projective geometry; and, in particular, it seemed highly desirable to seek some insight into the geometric construction and properties of systems of mutually Clifford parallel spaces. In this book we give such a development, essentially independent of that given by Wong, and consisting basically of a chapter in the projective geometry of complex projective space S_{2n-1} relative to a non-singular quadric primal Ω_{2n-2}. After some preliminaries (Chapter 2), we introduce the notion of Clifford parallel spaces with respect to Ω, using a straightforward generalization of Clifford's own definition (§ 3), and we rediscover Wong's additive sets (which we call Clifford reguli) and develop their properties (Chapter 3). This is followed by a mainly algebraic exposition of what we call *linear* systems of Clifford parallels (Chapter 4), analogous to, but rather more general than the systems discovered by Wong. The next Chapter is devoted to geometrical constructions for systems of Clifford parallels, particular attention being paid to the two most interesting cases, $n = 4$ and $n = 8$. The remainder of the book (Chapters 6 and 7) is concerned with representation theory; and this includes in particular a detailed account of a new sequence of algebraic varieties – here called T-models – which serve as birational maps of a specially significant kind of the pairs of polar $[n-1]$'s with respect to a quadric Ω in S_{2n-1}. Among the applications of our representation theory we encounter in Chapter 7 an apparently new construction for, and extended interpretation of, that well-known geometrical curiosity, the Study triality correspondence between the points and the two types of generating solids of a quadric Ω in S_7.

CHAPTER 2

PRELIMINARIES OF GEOMETRY IN S_{2n-1}

In this book we shall be concerned throughout, except when the contrary is stated, with geometry in a complex $(2n-1)$-dimensional projective space S_{2n-1} relative to a non-singular 'absolute' quadric primal Ω. We devote the present chapter, therefore, to recording some preliminary material about the appropriate coordinate systems, the properties of a quadric Ω, the definition and properties of the well-known type of algebraic variety which we call a *regulus* of $(n-1)$-dimensional spaces of S_{2n-1}, and some definitions and details relating to Grassmannian and Veronesean varieties.

1. Coordinate system in S_{2n-1}

In S_{2n-1} we shall generally use a system of allowable homogeneous coordinates

$$(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}) \tag{1.1}$$

based on a pair of skew $(n-1)$-dimensional reference spaces X, Y such that X contains the reference points $X_0, \ldots, X_{n-1}$ while Y contains $Y_0, \ldots, Y_{n-1}$. We shall write x (resp. y) for the column vector of the x_i (resp. y_i) and shall refer to (1.1) as the point (x, y). We note then that

(i) any $[n-1]$ of S_{2n-1} which is skew to the reference space Y has a matrix equation of the form

$$y = Ax, \tag{1.2}$$

where A is a uniquely defined $n \times n$ matrix of constants, and

(ii) the same $[n-1]$ is skew also to X if and only if $|A| \neq 0$. Two $[n-1]$'s given by $y = A_1x$ and $y = A_2x$ are skew to one another if and only if $|A_1 - A_2| \neq 0$.