HISTORICAL BACKGROUND

The germ of the theory of proximity spaces showed itself as early as 1908 at the mathematical congress in Bologna, when Riesz [95] discussed various ideas in his 'theory of enchainment' which have today become the basic concepts of the theory. The subject was essentially rediscovered in the early 1950's by Efremovič [18, 19] when he axiomatically characterized the proximity relation 'A is near B' for subsets A and B of any set X. The set X together with this relation was called an infinitesimal (proximity) space, and is a natural generalization of a metric space and of a topological group. A decade earlier a study was made by Krishna Murti [52], Wallace [116, 117] and Szymanski [113] concerning the use of 'separation of sets' as the primitive concept. In each case similar, but weaker, axioms than those of Efremovič were used. Efremovič later used proximity neighbourhoods to obtain an equivalent set of axioms for a proximity space and thereby an alternative approach to the theory.

Defining the closure of a subset A of X to be the collection of all points of X 'near' A, Efremovič [19] showed that a topology can be introduced in a proximity space and that one thereby obtains, in fact, a completely regular (and hence uniformizable) space. He further showed that every completely regular space X can be turned into a proximity space with the help of Urysohn's function: namely, $A \,\bar{\delta}\, B$ iff there exists a continuous function f mapping X into $[0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Smirnov [98] subsequently proved that every completely regular space has a maximal associated proximity space, and that it has a minimal associated proximity space if and only if it is locally compact.

More recently, Mrówka [75] has introduced a nearness relation on the set of all sequences from a proximity space. This provides a motivation for defining a notion, similar to that of proximity, in a Fréchet L-space; such a notion was discovered independently by Goetz [28] to obtain what he terms a $\mathscr{UL}$-space. However, as opposed to the complete regularity of a proximity space, a $\mathscr{UL}$-space need not even be a topological space. Poljakov [91] and Goetz [29] have since carried out further investigations in this area, and discuss the connection between this notion and the proximity of Efremovič. Švarc [20] had earlier introduced a nearness relation on the set of all nets from a given space.

In order to study mappings from one proximity space to another, it was natural for Efremovič to introduce the concept of a proximity mapping. Defined to be a mapping which preserves the proximity of sets, a proximity mapping is a natural analogue of a continuous mapping in topological spaces and of a uniformly continuous mapping in uniform spaces. It is readily verified that a proximity mapping between two proximity spaces is continuous with respect to the induced topologies. Pervin [86] later revealed that the converse holds if the domain proximity space is equinormal.

In 1952, Smirnov [98] pursued extensions of proximity spaces and in particular answered Alexandroff's query: 'which topological spaces admit a proximity relation compatible with the given topology?'. He discovered the connection between the Hausdorff compactification of a Tychonoff space and the compatible proximity relation, showing that 'a topological space admits a compatible proximity relation if and only if it is a subspace of a compact Hausdorff space'. Using the 'ends' of Alexandroff [A], Smirnov [98] obtained the compactification of a proximity space X by identifying each point $x \in X$ with the end consisting of all proximity neighbourhoods of x, and showing the compactification of X to be the set of all ends in X.

The concept of a cluster, the analogue in a proximity space of an ultrafilter, was introduced by Leader [53] and provides an alternative approach to many proximity problems. In particular, Leader [53] obtained the compactification of a proximity space X to be the family of all clusters from X. Since the Smirnov compactification is unique, it is evident that a one-to-one correspondence exists between clusters and ends. In fact, Leader [55] proves (without resorting to compactification theory) that clusters and ends are dual classes.

Császár and Mrówka [13] proved a stronger result regarding the compactification of a proximity space: namely, compactification can be effected preserving the proximity weight. This gave rise to the following interesting metrization theorem: a proximity space of proximity weight $\aleph_0$ is metrizable. Additional solutions to the metrization problem were offered earlier by Efremovič and Švarc [20], who used the 'sequence-uniformity' method, and by Ramm and Švarc [92] using uniform proximity covers. Smirnov [105] has used both the pseudo-metric and proximity neighbourhood approaches in deriving necessary and sufficient conditions for metrizability. More recently, Leader [61] has investigated this problem in a manner analogous to R. L. Moore's approach to the metrization of topological spaces.

Using the alternate set of axioms for uniform spaces involving uniform coverings, Smirnov [98] showed that every proximity (or p-) equivalence class of uniform structures contains a coarsest member, which is also the unique totally bounded structure of the class. He also showed that there is an isomorphism between the partially ordered set of all proximities on a given completely regular space and the partially ordered set of all its compactifications, which reduces the theory of proximity spaces to that of compactifications.

In 1959, Gál [27] continued the pursuit of the relationships between uniform structures and proximities, proving that there is a natural order-preserving one-to-one correspondence between totally bounded (precompact) structures and proximity relations. Gál also showed that there is a one-to-one correspondence between the compactifications of a uniformizable space and the totally bounded uniform structures which are compatible with its topology. Again, this yields a one-to-one map between the Hausdorff compactifications and the separated structures. In the same year, Alfsen and Fenstad [4] paralleled the work of Gál and treated many of the problems of Efremovič and Smirnov in the framework of Weil's uniform structures. Using maximal regular filters Alfsen and Fenstad perform completion and compactification, showing that there exists a maximal completion amongst all the completions determined by the structures of a given proximity equivalence class. This naturally raises the question as to whether there always exists a minimal completion, which is equivalent to asking if there exists a finest uniform structure in a given proximity equivalence class. This question was answered affirmatively by Smirnov [98] for the case in which the proximity space is metrizable. That the answer is in general negative was first shown in 1961 by Dowker [17]. He unveiled an example of a proximity space (a product of two infinite spaces with the product proximity structure) which has no finest uniform structure inducing its proximity.

The key concept in any completion theory for proximity spaces is that of 'small' sets, which can be introduced by means of pseudo-metrics, uniform structures or uniform coverings. Leader [54] has used the first device to define a local cluster, while the authors [121] have used the second to define a Cauchy cluster. They obtain a completion which consists of the family of all local (resp. Cauchy) clusters in X, a subspace of the family of all clusters, which Leader showed to be the Smirnov compactification. In [99, 100], Smirnov used uniform δ-coverings to introduce the concept of a complete uniform space and proved that every proximity space admits a minimal completion of this kind. Thus the general existence of a minimal completion was established, but with the sacrifice of the one-to-one correspondence between proximity structures and completions.

This was regained in the work of Alfsen and Njåstad [6], who in 1963 gave yet another example of a p-equivalence class lacking a finest uniform structure. (Further examples are to be found in Leader [56], Fenstad [23] and Isbell [44].) This then led to the notion of a generalized uniform structure, obtained from Weil's uniform structure by replacing the 'intersection' axiom with a less restrictive one. They proved that for generalized uniform structures the answer to Smirnov's problem is affirmative, and gave an explicit characterization of those generalized uniform structures which occur as the finest member of their respective p-equivalence classes. Such members are called *total* structures, and it was shown that the collection of all total generalized uniform structures embraces all ordinary metrizable (or pseudo-metrizable) uniform structures. Alfsen and Njåstad also found a relation between proximal continuity and uniform continuity, obtaining as a particular consequence, Efremovič's result that metric uniform continuity is equivalent to metric proximal continuity. They further proved that every generalized uniform space can be completed, and established a one-to-one correspondence between generalized uniform structures and proximity space completions. In particular, the minimal completion of a proximity space is obtained by completion of the finest generalized structure compatible with the proximity structure.

In the same year, Njåstad [80] further developed the concept of a generalized uniform structure. He first noted that the collection of all generalized uniform structures on a set and the collection of all proximity structures on a set form complete lattices when ordered by the relations 'finer-coarser', and that lattice sums and products are compatible, unlike the case in which the usual uniform structures are considered. Moreover, he established the existence and compatibility of initial (final) generalized uniform structures and initial (final) proximity structures. He proved that if the uniformity is replaced by the associated proximity, then uniform convergence implies convergence in proximity, a notion introduced by Leader [54]. Convergence in proximity implies uniform convergence if the associated total uniformity is used in place of the proximity. Moreover, for a net of functions with a linearly ordered directed set, the two forms of convergence are equivalent. In 1965, Hursch [37] formulated a new concept, *height*, to help clarify the order structure of p-equivalence classes of uniformities.

As previously mentioned, some authors have worked with weaker axioms than those of Efremovič, enabling them to introduce an arbitrary topology on the underlying set. With such generalized proximities as quasi-proximity, paraproximity, pseudo-proximity and local proximity already existing in the literature, one almost wonders if a generalized proximity relation may be defined for each prefix which can possibly be attached to the word 'proximity'! About 1963, both Pervin [84] and Leader [57] independently studied generalizations of Efremovič's original set of axioms. Pervin neglected the symmetry condition, obtaining what he called a quasi-proximity space. As well as omitting the symmetry condition, Leader used a weakened form of the 'Strong Axiom' to arrive at his topological d-space. It was shown that every topological space gives rise to a generalized proximity space (X, δ) of either form by defining the binary relation δ as follows: $A \delta B$ iff $A \cap \bar{B} \neq \varnothing$. Conversely, every quasi-proximity or topological d-space (X, δ) becomes a topological space if the closure operator is defined by $\mathrm{Cl}(A) = \{x : \{x\} \delta A\}$. Lodato [63] later added symmetry to Leader's set of axioms to obtain a symmetric binary relation which we shall refer to as a Lodato proximity. He proved that every set with a Lodato proximity defined on it satisfies the R_0 axiom (i.e. every open set contains the closure of each of its points), and that given any R_0-space we obtain a Lodato proximity compatible with the given topology if we define $A \delta B$ iff $\bar{A} \cap \bar{B} \neq \varnothing$. Mozzochi [72] has since introduced the idea of a symmetric generalized uniform structure and has studied its relationship to a Lodato proximity structure, as well as extending results of Alfsen–Fenstad, Hursch and others to such a setting.

In 1964, Hayashi [32] introduced the notion of 'paraproximity' by replacing the word 'finite' by 'arbitrary', and thereby strengthening Efremovič's 'union' axiom to read : for an arbitrary index set Λ, $(\bigcup_{\lambda \in \Lambda} A_\lambda) \delta B$ iff $A_\mu \delta B$ for some $\mu \in \Lambda$. He showed that a paraproximity space X is completely normal if one defines G to be an open set if and only if $G \bar{\delta} (X - G)$. A completely normal space becomes a paraproximity space if we define $A \delta B$ iff $A \cap \bar{B} \neq \varnothing$. Hayashi [33] also discussed a generalized proximity space, which he called a pseudo-proximity space, with even weaker axioms than those considered by Pervin or Leader. He proved that every such space can be topologized and that every topological space admits a pseudo-proximity.

Recently, Leader [59] has defined a local proximity space, in which both 'proximity' and 'boundedness' are taken as primitive terms. The proximity spaces of Efremovič are the special cases in which all subsets are bounded. Just as every proximity space can be embedded as a dense subset of a compact Hausdorff space, it is shown that every local proximity space can be embedded as a dense subset of a locally compact Hausdorff space. Leader also showed that every proximity space (X, δ) with its proximity relation localized with respect to any free regular filter from (X, δ) gives rise to a local proximity space. Conversely, every local proximity space arises from the localization of some proximity relation.

CHAPTER 1

BASIC PROPERTIES

1. Introduction

In a topological space X, the topology is determined by the closure axioms given by Kuratowski concerning the relation 'x is a closure point of $A \subset X$'. When x is a closure point of A, we may say that 'x is *near* A'. In terms of this nearness relation, a continuous function $f: X \to Y$ may then be described as one exhibiting the property: if x is near A, then $f(x)$ is near $f(A)$. This suggests axiomatizing the relation 'A is near B' for subsets A and B of X. For the case in which X is a pseudo-metric space with pseudo-metric d, this nearness relation can be defined in a natural way. Let

$$D(A, B) = \inf\{d(a, b): a \in A, b \in B\}.$$

We may then define:

$$A \text{ is near } B \quad \text{if and only if} \quad D(A, B) = 0.$$

In terms of D, the closure of a set A is $\bar{A} = \{x: D(A, x) = 0\}$. But the nearness relation so defined goes a little further. Let (Y, e) be another pseudo-metric space, E be defined in a similar manner to D, and f be a function from X to Y. Then f is uniformly continuous if and only if $D(A, B) = 0$ implies $E(f(A), f(B)) = 0$ (see (4.8)). Thus the nearness relation between the subsets is somehow connected with uniformity.

The above nearness relation (in a pseudo-metric space) satisfies the following properties, where we denote 'A is near B' by $A \delta B$:

(1.1) $$A \delta B \quad \text{implies} \quad B \delta A.$$

(1.2) $$(A \cup B) \delta C \quad \text{iff} \quad A \delta C \quad \text{or} \quad B \delta C.$$

(1.3) $$A \delta B \quad \text{implies} \quad A \neq \varnothing, \quad B \neq \varnothing.$$

(1.4) $A \not\delta B$ implies there exists a subset E such that $A \not\delta E$ and $(X - E) \not\delta B$.

(1.5) $$A \cap B \neq \varnothing \quad \text{implies} \quad A \delta B.$$

In a metric space, the nearness relation also satisfies:

(1.6) $x\,\delta\,y$ implies $x = y$.

(Strictly speaking one should use the notation $\{x\}\,\delta\,\{y\}$, but we shall simply write $x\,\delta\,y$.)

All of the above properties, except perhaps (1.4), are immediate consequences of the definition of the nearness relation. To verify (1.4), we note that if $A\,\bar{\delta}\,B$ then $D(A,B) = \epsilon > 0$. Setting

$$E = \{x \in X : D(x,B) \leqslant \epsilon/2\} \quad \text{we obtain} \quad D(A,E) \geqslant \epsilon/2$$

and $D(X-E,B) \geqslant \epsilon/2$, from which the desired property follows.

The above discussion leads to the following definition of a proximity space:

(1.7) Definition. *A binary relation δ on the power set of X is called an (Efremovič)* proximity *on X iff δ satisfies the axioms* (1.1)–(1.5). *The pair (X,δ) is called a* proximity space.

Proximity relations satisfying Axiom (1.6) will be referred to as *separated* (or *Hausdorff*) proximity relations. If a proximity is derived from a (pseudo-) metric, then it is called a *(pseudo-) metric proximity*.

(1.8) Remarks. The above axioms are different from, although equivalent to, the original axioms of Efremovič. The reason for writing the axioms in this way is to permit a smooth transition to generalized proximity spaces. It will be shown presently that a proximity δ on X induces a *topology* $\tau = \tau(\delta)$ on X if one defines the closure $\bar{A}$ of A to be the set $\{x : x\,\delta\,A\}$. It will be seen that this topology is always completely regular: in fact it is always Tychonoff if Axiom (1.6), which is equivalent to the T_1-axiom, is satisfied. Conversely, if (X,τ) is any completely regular topological space, then there exists a proximity δ on X such that $\tau(\delta) = \tau$.

Actually it would be sufficient to concern ourselves solely with separated proximity spaces, as many authors do; for if a given space fails to satisfy condition (1.6), we can instead consider the separated quotient space formed from the equivalence classes consisting of all points near to one another. However, for the sake of generality, we shall not assume a proximity space to be separated.

Suppose δ' is a binary relation on the power set of X that satisfies (1.2)–(1.5) and

(1.2′) $\quad A\,\delta'(B \cup C)$ iff $A\,\delta' B$ or $A\,\delta' C$.

Then δ' induces a topology $\tau(\delta')$ on X if one defines the closure $\bar{A}$ of A to be the set $\{x : x\,\delta' A\}$. If in addition we require that δ' satisfy the Symmetry Axiom (1.1), then the induced topology will be completely regular. Thus we see that the Symmetry Axiom (1.1) is in a sense equivalent to the complete regularity of the induced topology.

Axiom (1.4) plays an important role in the theory of proximity spaces, but is omitted or replaced by a weaker condition in some generalized proximity spaces. It will, therefore, be helpful to avoid the use of this axiom as far as possible. We shall refer to this axiom as 'the Strong Axiom.' It should be noted that the order of the sets in the Strong Axiom is important, particularly in generalized proximity spaces in which (1.1) is not satisfied. We shall strictly observe the order even in proximity spaces, so that the proofs can then be carried over to more general situations.

Given below is an outline of a number of examples of proximity spaces in which the proximity is not constructed from a pseudometric. Since some of these will be carefully taken up in later sections, the details are not verified here.

(1.9) EXAMPLE. Just as discrete and indiscrete topologies can be defined on any set, we have discrete and indiscrete proximities. If we define $A\,\delta_1 B$ iff $A \cap B \neq \varnothing$, then δ_1 is the *discrete proximity* on X. On the other hand, if $A\,\delta_2 B$ for every pair of non-empty subsets A and B of X, then we obtain the *indiscrete proximity* on X.

(1.10) EXAMPLE. Given a completely regular space (X, τ), we say that subsets A and B of X are *functionally distinguishable* iff there is a continuous function $f: X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

We may then define a proximity δ on X by

(1.11) $\quad A \not\delta B$ iff A and B are functionally distinguishable.

For details, refer to Theorem (2.10) and Remarks (3.15).

(1.12) EXAMPLE. Given a uniform space $(X, \mathscr{U})$, one may define a proximity on X by $A\,\delta\,B$ iff for every $U \in \mathscr{U}$, one (and hence all) of the three following equivalent conditions is satisfied:

(i) $U[A] \cap B \neq \varnothing$;
(ii) $A \cap U[B] \neq \varnothing$;
(iii) $(A \times B) \cap U \neq \varnothing$.

Several interesting results concerning the relationship between uniformities and proximities will be discussed in Chapter 3.

(1.13) EXAMPLE. If $(X, \cdot, \tau)$ is a topological group and $\mathscr{N}$ is the neighbourhood system of the identity, we may define

$$A\,\delta_1\,B \quad \text{iff for every} \quad N \in \mathscr{N}, \quad NA \cap B \neq \varnothing.$$

A second proximity δ_2 may be defined by

$$A\,\delta_2 B \quad \text{iff for every} \quad N \in \mathscr{N}, \quad AN \cap B \neq \varnothing.$$

In general the two proximities δ_1 and δ_2 differ. They coincide, however, if X is either commutative or compact.

2. Topology induced by a proximity

In this section we consider the topology on X which is induced by a proximity on X, and study its elementary properties. Properties (i) and (ii) of the following lemma, which follow directly from Axioms (1.1), (1.2) and (1.4), are useful in several proofs.

(2.1) LEMMA. (i) *If $A\,\delta\,B$, $A \subset C$ and $B \subset D$, then $C\,\delta\,D$. Hence X is near every non-empty subset.*

(ii) *If there exists an x such that $A\,\delta\,x$ and $x\,\delta\,B$, then $A\,\delta\,B$.*

Note that the Strong Axiom is not used in the proof of the following theorem.

(2.2) THEOREM. *If a subset A of a proximity space (X, δ) is defined to be* closed *iff $x\,\delta\,A$ implies $x \in A$, then the collection of complements of all closed sets so defined yields a topology $\tau = \tau(\delta)$ on X.*

Proof: Obviously $\varnothing$ and X are closed sets. Let $\{A_i : i \in I\}$ be an arbitrary collection of closed sets. If $x\,\delta \bigcap_{i \in I} A_i$ then by Lemma