# OPERATIONAL CALCULUS

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## BASED ON THE TWO-SIDED LAPLACE INTEGRAL

BY

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'La langue de l'analyse, la plus parfaite de toutes les langues, étant par elle-même un puissant instrument de découvertes; ses notations, lorsqu' elles sont nécessaires et heureusement imaginées, sont les germes de nouveaux calculs.'

LAPLACE, *Théorie analytique des probabilités*, Paris, 1812, p. 7

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# PREFACE

This book was developed from a series of lectures [by van der Pol] given at the 'Technische Hogeschool' of Delft during 1938 and following years, and from a second series given during the first half of 1940 at the 'Philips Research Laboratories', Eindhoven.

The second author [Bremmer] made extensive lecture reports on the latter series; subsequently the material was jointly extended during the German occupation of the Netherlands. In this period the original manuscript, in Dutch, was practically completed, while several problems founded on it were published from 1940 on in *Wiskundige Opgaven* of the Netherlands 'Wiskundig Genootschap'. The English translation of the original manuscript was edited by Dr C. J. Bouwkamp, of this Laboratory.

Primarily this book is intended for application of the operational calculus in its modern form to mathematics, physics and technical problems. We have therefore given not only the basic principles, ideas and theorems as clearly as possible (and rather extensively), but also many worked-out problems from purely mathematical and physical as well as from technical fields. In order to limit the size of the book, proofs of some of the deeper theorems have been omitted, and for these the reader is referred to the mathematical literature.

None the less, it is believed that the purely mathematical treatment is more advanced than is usual in books devoted primarily to practical applications. The Abel and Tauber theorems, for example, are extensively considered, with many examples taken from pure mathematics as well as from technical problems.

It is therefore hoped that the book may be of value to those pure mathematicians who are interested in a rapid and simple derivation of complicated and unexpected relations between various mathematical functions, as well as to the engineer in search, for example, of a very simple treatment of transient phenomena in electrical networks, such as filters. In both cases the operational calculus appears to its best advantage.

Furthermore, several applications of the operational calculus to the theory of numbers are to be found in this book, a field of application which appears to be of the greatest heuristic value and which at present seems to be far from exhausted.

We have endeavoured to give the operational calculus a rigorous mathematical basis; on the other hand, we have tried to give the subject-matter such a form that it can be applied simply to practical problems.

This led us to treat the operational calculus *ab initio*, by means of the bilateral or two-sided Laplace integral, contrary to the usual practice based on the one-sided Laplace integral. This procedure was greatly stimulated by extensive discussions with Dr Ph. le Corbeiller, now at Harvard University, Cambridge, Mass.

The foundation of the theory on the two-sided Laplace transform caused us to introduce at an early stage:

(*a*) the Heaviside unit function, $U(t)$, defined by $U(t) = 0$ for $t < 0$, $\frac{1}{2}$ for $t = 0$, and 1 for $t > 0$,

(*b*) the Dirac $\delta$-function, $\delta(t)$.

Further, the use of the two-sided Laplace transform requires, for each operational relation, the stipulation of the band of $\mathrm{Re}\, p$ within which this relation is valid. It is felt, however, that the latter complication is more than compensated for by the following advantages:

(i) the class of functions suited to an operational treatment becomes much larger,

(ii) the 'transformation rules' are considerably simplified,

(iii) the entire treatment becomes more rigorous than the usual presentation of the one-sided integral in technical books.

The rapid way in which solutions of complicated problems can be found with operational calculus is often astounding. This is mainly due to the fact that discontinuous functions $h(t)$ of a real variable $t$, which frequently occur in the treatment of electrical and mechanical transients as well as in the theory of numbers, have an operational 'image' $f(p)$ that is analytic in some band of finite or infinite width of the complex $p$-plane. Simple transformations of these smooth, analytic, functions $f(p)$ then correspond uniquely to operations on the discontinuous functions $h(t)$, and so the complicated handling of these discontinuous functions can be replaced by extremely simple transformations of the corresponding analytic functions.

The treatment in this book is not limited to the Laplace transform of functions of one single variable; an extensive chapter is also devoted to the multidimensional Laplace transforms. This 'simultaneous operational calculus' enabled us, for example, to treat Green's functions in potential and wave problems; this part was mainly developed by the second author. Thus familiar solutions of the Maxwellian equations are obtained by a few extremely simple algebraic transformations in the $p$-field.

In the Introduction the reader will find a summary of the subjects treated in the various chapters.

The authors wish to express their thanks to several friends who read parts or the whole of the manuscript. In the first place we wish to thank Dr C. J. Bouwkamp for many remarks, which, we believe, have improved

the clarity of the exposition, and for the translation of the original manuscript. Further we wish to thank Prof. N. G. de Bruÿn, whose remarks have contributed materially to the rigour of the treatment.

In conclusion, we would add that we shall be most grateful for remarks and criticism from readers. We feel that this first treatment of the practical operational calculus on the basis of the two-sided Laplace transform, with so many applications to both pure and applied mathematics, and a very great part of which we believe to be essentially new, is bound to show marks of immaturity typical of young scientists as well as of young sciences.

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*November* 1947

## PREFACE TO THE SECOND EDITION

The necessity of a second edition, four years after the appearance of the first one, gave us the opportunity to insert a number of corrections and improvements, scattered throughout the book. The greater part of the corrections are due to suggestions by several correspondents. In connexion herewith we wish to thank particularly Prof. S. Colombo (Paris); Prof. A. Erdelyi (Pasadena); Dr J. H. Pearce (London); and Mr H. van de Weg (Eindhoven).

We have also inserted the following new paragraphs:

Rules for the treatment of *correlation functions*,
A note on the theory of *distributions*,
A note on the *Wiener-Hopf technique*,

as all three, modern, subjects lend themselves well to a concise treatment in the 'language' of operational calculus.

Finally we wish to thank the Cambridge University Press for the care again shown in the preparation of this second edition.

B. v. D. P.
H. B.

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