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CHAPTER I

GENERAL INTRODUCTION

1. History of the operational calculus

The operational calculus, a modern treatment of which is aimed at in the present book, can be traced back as far as the work of Oliver Heaviside (1850–1925). Though many scientists (Leibniz, Lagrange, Cauchy, Laplace, Boole, Riemann, and others) preceded Heaviside in introducing operational methods into analysis†, a systematic use of it in physical and technical problems was stimulated only by Heaviside's work.

Heaviside‡ was a 'self-made man', deprived of regular study at the university or the engineering college. Nevertheless, his curious methods, created by himself as they often were, led him to results in technics and theoretical physics that are undoubtedly among the most important ever reached. In this connexion let us remember that Heaviside's work § already contains Maxwell's equations of the electromagnetic field in the modern, now current, vector notation. Also due to him is the conception of the 'Heaviside Layer', which is of the greatest importance in present-day radio communication. Moreover, independently of Lorentz, Heaviside enunciated the theory of the electronic motion in a magnetic field; he further introduced into Maxwell's theory that part of the total current which is due to convection. His concept of impedance, defined independently of Kennelly, is much more general than that of the conventional alternating-current technique. The notion of 'negative resistance', now common property in electrical engineering (e.g. arc lamp, radio valve), is often put forward in his papers, and for the first time in 1895.

But it may be stated that even to-day Heaviside's papers, difficult to read as perhaps they are, still contain a great many views and hidden things, of both mathematical and physical interest, which are not yet very well known and which, therefore, have not met with proper appreciation. Certainly this is largely due to the strange manner in which Heaviside often derives and announces his results. Moreover, the fact that Heaviside was not a university man raised a barrier, a certain antagonism, between him and his contemporaries. The latter reproached him, rightly, with his great

† Compare, for instance, H. T. Davis, *The Theory of Linear Operators*, Bloomington, Indiana, 1936.

‡ For a survey of the life and work of Heaviside the reader is referred to E. T. Whittaker, *Bull. Calcutta Math. Soc.* XX, 216, 1928–9; Balth. van der Pol, *Ned. Tijdschr. Natuurkunde* V, 269, 1938.

§ O. Heaviside, *Electrical Papers*, vols. I and II, Macmillan, London (New York), 1892; *Electromagnetic Theory*, vols. I, II and III (1893–1912), reissued 1922 by Benn Brothers, London.

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lack of mathematical rigour. Yet Heaviside did develop an abundance of mathematical and physical methods and results which afterwards, on critical elaboration by various scientists, proved to be substantially true and have been approved as such. Though perhaps reasonable, it is regrettable that such a barrier existed between Heaviside and his fellow-mathematicians. Equally regrettable, but certainly unreasonable, is the point of view occasionally taken by modern mathematicians with regard to Heaviside's work; in many respects it is far superior to the later contributions to this part of science, both for the methods as well as for the results arrived at†.

Fortunately, there are other records too. For instance, Whittaker (loc. cit.) wrote, after discussing the difference in views on mathematics between Heaviside and the pure mathematician:

'Looking back on the controversy after thirty years, we should now place the Operational Calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the Tensor Calculus as the three most important mathematical advances of the last quarter of the nineteenth century. Applications, extensions and justifications of it constitute a considerable part of the mathematical activity of to-day.'

It is this Operational Calculus to which the present book is devoted.

2. The operational calculus based on the Laplace transform

Heaviside's ideas concerning the operational calculus may perhaps best be interpreted as follows‡. Imagine a linear electrical network originally at rest. Let an electromotive force $E(t)$ be applied to it, where $E(t)$ is an arbitrary function of the time t. The response current, $i(t)$, is then determined by

$$i(t) = Y(D_t)\,E(t), \tag{1}$$

in which $D_t = d/dt$. The function $Y(D_t)$ is an *operator function* applied to the *operand* $E(t)$, to give the current $i(t)$. If $E(t)$ is constant with time, $i(t)$ will be constant too; under these circumstances $Y(D_t)$ degenerates into the reciprocal of an ohmic resistance.

The question arises at once of how we are to interpret the operator function when, for instance, it is of the following form:

$$Y(D_t) = \frac{1}{1+D_t}.$$

† Heaviside was more than 'ein englischer Elektroingenieur', in spite of his (and his successor's) methods being 'mathematisch sehr unzulänglich' and 'allerdings mathematisch unzureichend'. Quotations from G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937, and New York, 1943, pp. 337, 421.

‡ *Proc. Roy. Soc.* LII, 504, 1892–3; LIV, 105, 1893.

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Should we take it in the sense of

$$\frac{1}{1+D_t} = 1 - D_t + D_t^2 - D_t^3 + \ldots, \tag{2}$$

or rather in the sense of

$$\frac{1}{1+D_t} = \frac{1}{D_t} - \frac{1}{D_t^2} + \frac{1}{D_t^3} - \ldots ? \tag{3}$$

In the first case it is reasonable to interpret D_t^n as d^n/dt^n. Similarly, when applying (3) we would take $1/D_t$ to mean $\int_a^t d\tau$; and in so doing a second question has presented itself: What value of the constant of integration, a, is required? Guided by practical experience, Heaviside came to the conclusion that, if possible, the form (3) should be chosen rather than (2). He further concluded that in discussing switch-on phenomena in electrical networks (when the electromotive force does not come into action before $t = 0$; that is, $E(t) = 0$ when $t < 0$), the lower limit of integration, a, has to be equated to zero. However, in Heaviside's work, we have not been able to find any rigorous statements concerning this question in general.

A modern treatment of the operational calculus requires, therefore, a much more rigorous base. This is furnished by the Laplace transform, as was already pointed out by Heaviside himself†, though he did not use it extensively. The same Laplace transform was the starting-point of later writers such as Carson‡, Bush§, Humbert‖, Doetsch¶, Wagner††, Droste‡‡, McLachlan§§, and Widder‖‖. When $h(t)$ is supposed to be given, then the Laplace transform of $h(t)$ is the function $f(p)$, defined by the following integral:

$$f(p) = p \int_0^\infty e^{-pt} h(t)\, dt. \tag{4}$$

By so doing we let the function f of the variable p correspond to the function h of the variable t. Conversely, as particularly discussed by Carson, the formula (4) may be considered as an integral equation for the unknown function $h(t)$, when $f(p)$ is supposed to be given.

† *Electromagnetic Theory*, III, 236.

‡ John R. Carson, *Electric Circuit Theory and the Operational Calculus*, New York, 1926.

§ V. Bush, *Operational Circuit Analysis*, New York, 1929.

‖ P. Humbert, *Le calcul symbolique*, Paris, 1934.

¶ G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937, and New York, 1943.

†† K. W. Wagner, *Operatoren Rechnung*, Leipzig, 1940.

‡‡ H. W. Droste, *Die Lösung angewandter Differentialgleichungen mittels Laplacescher Transformation*, Berlin, 1939.

§§ N. W. McLachlan, *Complex Variable and Operational Calculus*, Cambridge, 1939.

‖‖ D. V. Widder, *The Laplace Transform*, Princeton, 1941.

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A somewhat different point of view is taken by Bromwich†, who started from the complex integral

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\frac{f(p)}{p}\,dp; \tag{5}$$

this integral, it may be noted in passing, was known to Riemann‡ as early as 1859. A complete survey of Bromwich's work is to be found in Jeffreys' book §. Wagner|| also based his contribution upon the integral (5). Further impact to the calculus is owed to Lévy¶, who pointed out that the solution of (4), considered as an integral equation for $h(t)$, is given by (5), and vice versa. Thus by Lévy's work the two different points of view came together in one consistent theory.

Also based on the Laplace transform (4), with zero as lower limit of integration, are the former investigations of Van der Pol†† and of Van der Pol and Niessen‡‡.

Henceforth the transformation (4) will be called the unilateral or *one-sided* Laplace transform. Contrary to the earlier investigations, this book will be based on the *two-sided* Laplace transform

$$f(p) = p\int_{-\infty}^{\infty} e^{-pt}h(t)\,dt, \tag{6}$$

to obtain a wider base for the operational calculus, as will be discussed in detail in the next chapter. The two-sided Laplace integral has its lower limit of integration equal to $-\infty$ instead of 0. This generalization proves very advantageous, and includes the earlier calculus as a special case. In the first place, the operational rules are considerably simplified by the generalization and, secondly, a much larger class of functions (and phenomena) becomes accessible.

It is worth while to remark that, whether we use (4), (5) or (6) as the basis of the operational calculus, the indefinite concepts of operator and operand wholly disappear. Instead of the vague formulation of the early operational calculus there comes the functional transform (6), by which there corresponds to any given function $h(t)$ a new function $f(p)$ of the complex variable p. In the Volterra sense, $f(p)$ is a 'fonction de ligne' or 'fonctionnelle', indicating that the form of the function $f(p)$ depends on the

† T. J. I'a. Bromwich, *Proc. Lond. Math. Soc.* xv, 401, 1916.

‡ The integral occurs in Riemann's classical paper of only eight pages: 'Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse', *Monatsber. Berl. Akad.* Nov. 1859; see also *Gesammelte Werke*, Leipzig, 1876, p. 136.

§ H. Jeffreys, *Operational Methods in Mathematical Physics*, Cambridge, 1927.

|| K. W. Wagner, *Arch. Elektrotech.* IV, 159, 1916.

¶ P. Lévy, *Le calcul symbolique d'Heaviside*, Paris, 1926.

†† Balth. van der Pol, *Phil. Mag.* VII, 1153, 1929; VIII, 861, 1929; XXVI, 921, 1938; *Physica,'s-Grav.*, IV, 585, 1937.

‡‡ Balth. van der Pol and K. F. Niessen, *Phil. Mag.* XI, 368, 1931; XIII, 537, 1932.

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whole set of values which $h(t)$ assumes on the complete real axis of t, $-\infty < t < \infty$.

It is to be emphasized that (6) is essentially a *linear* functional transform, since in the integrand of (6) the function $h(t)$ occurs linearly. As a consequence, the operational calculus is applicable only to linear problems such as switch-on phenomena in linear networks, problems of small vibrations, heat diffusion, potential theory, and electrical cables.

As far as the general outlines of the theory are concerned, this book is restricted to giving an extensive survey; the more complicated theorems underlying the theory are usually stated without proof. For proofs the reader is always referred to existing literature, cited in the text. Our main aim is to demonstrate the vigour of the operational calculus in its applications, by giving many examples. The discussion will not be confined to applications in physics and technics; many problems of pure mathematics will be included too.

If the reader has made himself familiar with the fundamental principles of the calculus presented here, he will certainly become aware of the strength of this mathematical tool of almost unrestricted heuristic-analytic value; he will be guided by many examples illustrating the general theory; he will often be able to construct new analytic relations by quite simple means.

3. Survey of the subject-matter

The starting-point of chapter II is the Fourier integral, on which the foundation of the operational calculus is built. We are then led back to the fundamental expressions (5) and (6). In chapter III some elementary 'operational relations' are derived which prove useful in the course of the subsequent investigations. In chapter IV we shall establish elementary 'operational rules', indicating how certain changes of the p-function correspond to others of the t-function. Chapter V is devoted to a detailed discussion of the unit function, $U(t)$, and the delta or impulse function, $\delta(t)$. The latter was introduced by Dirac in quantum mechanics; but Heaviside had already used it extensively before him. The impulse function is particularly important in relation to the Green function of differential equations. It is formulated in terms of the general concept of the Stieltjes integral. Chapter VI should be considered as a deepening and extension of chapter II. It contains a detailed investigation of questions of convergence, particularly in connexion with the summing of series and integrals by the well-known methods of Abel and Cesàro. In chapter VII, especially the asymptotic expressions for 'image' and 'original' are outlined, as well as related topics. Further, chapter VIII concerns the operational treatment of differential equations having constant coefficients. This matter is extended to a system of equations in chapter IX. These two chapters also include

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the theory of linear electrical networks, together with the corresponding transient phenomena. Differential equations with variable coefficients are treated in chapter X. Applications are made to Legendre polynomials, Bessel functions, etc. The matter of chapter XI must be considered as a generalization of that given in chapter IV; general rules of more complicated character are discussed. Chapter XII is devoted to the study of step functions, with applications, amongst others, to number-theoretic functions. In chapters XIII and XIV we consider the operational calculus applied to difference equations and integral equations respectively. Chapters XV and XVI concern applications of the theory to problems in several independent variables, particularly with respect to linear partial differential equations. Chapter XVI is thereby based on the simultaneous transposition of more than one variable, which leads to the *simultaneous operational calculus*.

It is clear from the survey given above that the subject-matter in any chapter is determined by some specific part of mathematics to which the operational calculus is successfully applicable in one way or another. It may thus happen that closely interrelated 'operational rules', on the one hand, and 'operational relations' concerning some definite type of function, on the other, are discussed at several places scattered through the book. This may hamper further applications, and the material presented must therefore be made more readily available. We have done this by listing the most important results in an appendix at the end of the book. We have thus an opportunity to supply the reader with some additional results which have not been given explicitly in the course of the work. The first list contains the 'operational rules'; it forms the 'grammar' of the operational calculus. The second list is the 'dictionary', helpful in translating the language of t into that of p and vice versa. The 'operational relations' are ordered so that those which concern related functions are grouped together.

The division indicated is such that some chapters and sections may be omitted by those who are interested in the applications of the operational calculus to technical problems only. Similarly, other chapters may be omitted by the pure mathematician.

The practical man will find in the following chapters and sections an almost complete course for his purpose:

II, §§ 5, 6, 7; III, §§ 1, 2, 3, 5, 6; IV, except § 7; V, except §§ 4, 5, 7; VI, § 3; VII, except §§ 5, 12; VIII; IX; X, §§ 1, 2, 4; XI, except § 5; XII, §§ 1, 2, 3; XIV, §§ 1, 2, 3, 5; XV; XVI.

On the other hand, the mathematician will find a survey of the subjects of interest to him by reading only the following chapters and sections:

II; III; IV; V; VI; VII; VIII, §§ 1, 2, 3, 4; X; XI; XII; XIII; XIV; XV, except § 4; XVI.

Both may well find it helpful to read the complete text, since the parts otherwise omitted will serve to throw light on the recommended selection.

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CHAPTER II

THE FOURIER INTEGRAL AS BASIS OF THE OPERATIONAL CALCULUS

1. The Fourier integral

The fundamental formulae of the operational calculus as developed in this book are closely connected with the Fourier integral, the theory of which will be recalled in this chapter. In subsequent chapters the usefulness of the operational calculus is demonstrated by means of simple examples and rules, and then, in chapters VI and VII, the foundations of the operational calculus are studied once more and in greater detail. Since the fundamental principles will be discussed first, inasmuch as they are directly connected with the Fourier integral, and since the Fourier integral may be considered as the limiting case of the Fourier series, we shall start with a short account of the latter.

It is well known that every periodic function—subject to proper conditions, which are always satisfied in applications to physical problems—can be expanded into an infinite series of trigonometric functions, viz.

$$h(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) + \sum_{n=0}^{\infty} b_n \sin(n\omega t),$$

in which $2\pi/\omega$ denotes the period of the function $h(t)$ and ω is the so-called *angular frequency*. Replacing the sine and cosine functions by their exponential equivalents we obtain just one simple Fourier series:

$$h(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\omega t}, \tag{1}$$

where the coefficients c_n are to be determined from

$$c_n = \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} e^{-in\omega\tau} h(\tau) \, d\tau.$$

The Fourier expansion (1) indicates that a single periodic function of frequency ω is virtually composed of an infinite number of trigonometric functions of angular frequencies ω, 2ω, 3ω, ..., respectively. As the original period increases (thus $\omega \to 0$) these frequencies get nearer and nearer to one another, whilst the individual coefficients c_n tend to zero as well. Therefore, in the limit, the Fourier series (1) is transformed into the *Fourier integral*, which reads as follows:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau} h(\tau). \tag{2}$$

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Provided that in (2) the integration with respect to the variable ω be taken in the sense of Cauchy's 'valeur principale' (principal value), i.e.

$$\int_{-\infty}^{\infty} d\omega = \operatorname{Lim}_{\lambda\to\infty}\int_{-\lambda}^{\lambda} d\omega,$$

then a rigorous investigation† of the Fourier integral (2)—often called *Fourier identity*—shows that the following system of (sufficient) conditions guarantees the validity of the Fourier identity at $t = t_0$:

(1) $h(t)$ has limited total fluctuation in the neighbourhood of $t = t_0$;

(2) $h(t)$ is integrable in any finite interval; moreover, the integral $\int_{-\infty}^{\infty} |h(t)|\, dt$ exists;

(3) at points of discontinuity the value of the function $h(t)$ is equal to the corresponding *mean* value, thus (see fig. 1)

$$h(t_0) = \operatorname{Lim}_{\epsilon\to 0}\frac{h(t_0+\epsilon)+h(t_0-\epsilon)}{2}. \qquad (3)$$

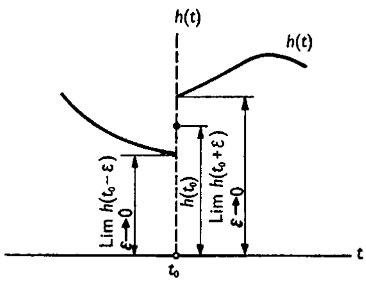

Fig. 1. Mean value of a discontinuous function.

The conditions quoted above are less stringent than the well-known earlier conditions of Dirichlet, and this is made possible by the introduction of the modern concept of 'limited total fluctuation'. As to the definition of this important concept, the function $h(t)$ is said to have limited total fluctuation in the interval $a < t < b$ if, for *every* set of subdivisions according to

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b,$$

the sum $\sum_{m=1}^{n} |h(t_m) - h(t_{m-1})|$ has an upper bound K, independent of n. The existence of the mean value as referred to in condition (3) follows from the fact that $h(t)$ has limited total fluctuation in an (arbitrarily) small interval around $t = t_0$. It should also be noted that, as regards the validity of the Fourier identity (2) at some definite point $t = t_0$, only the behaviour of $h(t)$ in the vicinity of $t = t_0$ is decisive. On the other hand, if $h(t)$ has limited total fluctuation in *any* finite interval, then its integrability follows at once. Moreover, any function with limited total fluctuation can be considered as the sum of two monotonic functions, one of which is non-decreasing, the other non-increasing. It is important to notice that, with respect to the validity of the Fourier identity, $h(t)$ need not be a *periodic* function, as was required in the analogous Fourier-series expansion. Moreover, we

† For details the reader is referred to E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937; S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932.

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would emphasize that the conditions we have mentioned are only of a sufficient character; they are by no means necessary, and there do exist less restrictive conditions, especially if the Fourier integral be taken in the sense of the Cesàro or Cauchy limits (cf. VI, §9). Finally, we may remark that the complex expression (2) can also be written in the equivalent real form

$$h(t) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty d\tau \cos\{\omega(t-\tau)\}\, h(\tau).$$

Example. Consider the function $h(t)$ defined by $(a>0)$

$$h(t) = \begin{cases} 0 & \text{if} \quad t<-a, \\ b & \text{if} \quad -a<t<a, \\ 0 & \text{if} \quad t>a. \end{cases}$$

For obvious reasons (see fig. 2) this function is called the *rectangle function*. From (2) its Fourier integral follows readily:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^\infty d\omega\, e^{i\omega t}\, b\int_{-a}^{a} d\tau\, e^{-i\omega\tau} = \frac{b}{\pi}\int_{-\infty}^\infty e^{i\omega t}\frac{\sin(\omega a)}{\omega}\, d\omega. \tag{4}$$

This integral represents the rectangle function in the complete interval $-\infty<t<\infty$, including the points of discontinuity, if, and only if, the rectangle function is defined to assume the mean value $\frac{1}{2}b$ at $t=\pm a$. The real Fourier integral of the rectangle function is given by

$$h(t) = \frac{2b}{\pi}\int_0^\infty \sin(\omega a)\cos(\omega t)\frac{d\omega}{\omega}.$$

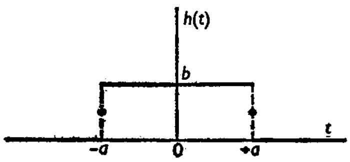

Fig. 2. The rectangle function.

2. A pair of integrals equivalent to the Fourier identity

In physical applications the variables t and ω often denote the time and the angular frequency, respectively. Then the Fourier integral gives the frequency spectrum (or spectral composition) of the original t-function; for, if the second integral in the Fourier identity is considered as a new function, viz.

$$g(\omega) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^\infty e^{-i\omega\tau}\, h(\tau)\, d\tau, \tag{5a}$$

then the Fourier integral itself can be written as follows:

$$h(t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^\infty e^{i\omega t}\, g(\omega)\, d\omega. \tag{5b}$$

Formula (5b) indicates that $h(t)$ may be looked upon as a continuous sum of periodic components $e^{i\omega t}$, the strength of the individual components being proportional to $g(\omega)$. Actually $\frac{1}{\sqrt{(2\pi)}}g(\omega)\, d\omega$ represents the total strength of all those components which lie inside the infinitesimal frequency

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band ω, $\omega + d\omega$. In general $g(\omega)$ is a complex function of the real variable ω; therefore one may write

$$g(\omega) = A(\omega)\, e^{i\phi(\omega)},$$

in which the modulus $A(\omega)$ and the argument $\phi(\omega)$ are both real functions of the frequency. Substituting this in (5b) we obtain

$$h(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} A(\omega)\, e^{i\{\omega t + \phi(\omega)\}}\, d\omega,$$

from which it is evident that $\frac{1}{\sqrt{(2\pi)}} A(\omega)\, d\omega$ represents the total amplitude of all vibrations that are contained in $(\omega, \omega + d\omega)$ and have a common phase of amount $\phi(\omega)$ at the time $t = 0$. Summarizing, we may say that both amplitude and phase of the spectral components of $h(t)$ are completely determined by the corresponding frequency function $g(\omega)$.

Example. Returning to the rectangle function, we obtain from (4)

$$g(\omega) = b \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin(\omega a)}{\omega}. \tag{6}$$

In this case $\phi(\omega)$ vanishes identically because $g(\omega)$ is real; hence the spectral components are equally phased at $t = 0$.

The pair of integrals (5), which is completely equivalent to the Fourier identity (2), reveals a striking symmetry with respect to the variables ω and t on the one hand, and the functions $g(\omega)$ and $h(t)$ on the other. Indeed, formula (5b) can be obtained from that of (5a) by interchanging ω and t, and then substituting $h(x)$ for $g(x)$ and $g(-x)$ for $h(x)$, and vice versa. Therefore, there also exists a typical symmetry in physics between any time function and its spectral composition. This principle of duality may be illustrated by the rectangle function treated above. The counterpart of this time function is the frequency function $g(\omega)$ that has a non-vanishing constant amplitude inside, and a zero amplitude outside, some finite ω-interval. It should be noticed that this type of spectrum is well known in the theory of electrical filters (frequency response). Since the rectangle function has a spectrum given by $\frac{\sin(a\omega)}{\omega}$ it is obvious that the time function belonging to a frequency band of width $2a$ is proportional to $\frac{\sin(at)}{t}$.

Another remarkable feature of (5) is that the *kernel* $e^{\mp i\omega t}$ contains the variables ω and t only in the combination of their product, ωt. There exist, of course, other systems of integral transforms showing the same peculiarity; for instance,

$$g_1(\omega) = \int_0^{\infty} J_\nu\{2\sqrt{(\omega t)}\}\, h_1(t)\, dt \quad (\omega > 0), \tag{7a}$$

$$h_1(t) = \int_0^{\infty} J_\nu\{2\sqrt{(\omega t)}\}\, g_1(\omega)\, d\omega \quad (t > 0), \tag{7b}$$

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