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REPRESENTATION**

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**BY
C. CARATHÉODORY**

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NOTE BY THE GENERAL EDITOR

PROFESSOR Carathéodory made the few corrections necessary in the text of the first edition of this tract, completed the manuscript of the new Chapter VIII, compiled the Bibliographical Notes afresh, and wrote the Preface to the second edition, during the later months of 1949. At his request, Mr G. E. H. Reuter, of the University of Manchester, agreed to revise the author's English. This work was completed shortly before Professor Carathéodory's death on 2 February 1950. Mr Reuter then kindly undertook the task of reading the proofs and attended to all points of detail which arose while the tract was going through the Press.

W. V. D. H.

CAMBRIDGE
January 1951

PREFACE TO SECOND EDITION

THIS reprint of my tract is almost without change, save for the addition of a chapter on the celebrated theorem of Poincaré and Koebe on uniformisation. If I have succeeded in making this chapter rather short, it is because I have been able to avail myself of the beautiful proof of van der Waerden (30), which has enabled me to expound the topological side of the problem in a few pages.

C. CARATHÉODORY

MUNICH
December 1949

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PREFACE TO THE FIRST EDITION

THIS little book is the outcome of lectures which I have given at various times and at different places (Göttingen, Berlin, Athens, Munich, and at the University of Harvard). It contains the theory of conformal representation as it has developed during the last two decades. The first half of the book deals with some elementary subjects, knowledge of which is essential for the understanding of the general theory. The exposition of this theory in the last three chapters uses the simplest methods available to-day.

The original manuscript, written in German, has been translated by Mr B. M. Wilson of the University of Liverpool and by Miss Margaret Kennedy of Newnham College. I wish to express here my warmest thanks for the care they have taken that the most intricate arguments should be made clear to the reader. I am also indebted to Prof. Erhard Schmidt (Berlin) and to Prof. Tibor Radó (Columbus, Ohio) for various improvements in the mathematical demonstrations, and to Miss Kennedy for several suggestions that simplified the text. Finally, my thanks are due to the staff of the Cambridge University Press for the admirable way in which their part of the work has been carried out.

C. CARATHÉODORY

ATHENS
December 1931

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