INTRODUCTION

HISTORICAL SUMMARY

1. By an isogonal (*winkeltreu*) representation of two areas on one another we mean a one-one, continuous, and continuously differentiable representation of the areas, which is such that two curves of the first area which intersect at an angle α are transformed into two curves intersecting at the same angle α. If the sense of rotation of a tangent is preserved, an isogonal transformation is called *conformal*.

Disregarding as trivial the Euclidean magnification (*Ähnlichkeitstransformation*) of the plane, we may say that the oldest known transformation of this kind is the *stereographic projection* of the sphere, which was used by *Ptolemy* (flourished in the second quarter of the second century; died after A.D. 161) for the representation of the celestial sphere; it transforms the sphere conformally into a plane. A quite different conformal representation of the sphere on a plane area is given by *Mercator's Projection*; in this the spherical earth, cut along a meridian circle, is conformally represented on a plane strip. The first map constructed by this transformation was published by *Mercator* (1512–1594) in 1568, and the method has been universally adopted for the construction of sea-maps.

2. A comparison of two maps of the same country, one constructed by stereographic projection of the spherical earth and the other by Mercator's Projection, will show that conformal transformation does not imply similarity of corresponding figures. Other non-trivial conformal representations of a plane area on a second plane area are obtained by comparing the various stereographic projections of the spherical earth which correspond to different positions of the centre of projection on the earth's surface. It was considerations such as these which led *Lagrange* (1736–1813) in 1779 to obtain all conformal representations of a portion of the earth's surface on a plane area wherein all circles of latitude and of longitude are represented by circular arcs (1).

3. In 1822 *Gauss* (1777–1855) stated and completely solved the general problem of finding *all* conformal transformations which transform a sufficiently small neighbourhood of a point on an arbitrary analytic surface into a plane area (2). This work of Gauss appeared to give the whole inquiry its final solution; actually it left unanswered the much more difficult question whether and in what way a given finite portion of the surface can be represented on a portion of the plane. This was first pointed out by *Riemann* (1826–1866), whose Dissertation (1851) marks a turning-point in the history of the problem which has been decisive for its whole later development; Riemann not only introduced all the ideas which have been at the basis of all subsequent investigation of the problem of conformal representation, but also showed that the problem itself is of fundamental importance for the theory of functions (3).

4. Riemann enunciated, among other results, the theorem that every simply-connected plane area which does not comprise the whole plane can be represented conformally on the interior of a circle. In the proof of this theorem, which forms the foundation of the whole theory, he assumes as obvious that a certain problem in the calculus of variations possesses a solution, and this assumption, as *Weierstrass* (1815–1897) first pointed out, invalidates his proof. Quite simple, analytic, and in every way regular problems in the calculus of variations are now known which do not always possess solutions (4). Nevertheless, about fifty years after Riemann, *Hilbert* was able to prove rigorously that the particular problem which arose in Riemann's work does possess a solution; this theorem is known as *Dirichlet's Principle* (5).

Meanwhile, however, the truth of Riemann's conclusions had been established in a rigorous manner by *C. Neumann* and, in particular, by *H. A. Schwarz* (6). The theory which Schwarz created for this purpose is particularly elegant, interesting and instructive; it is, however, somewhat intricate, and uses a number of theorems from the theory of the logarithmic potential, proofs of which must be included in any complete account of the method. During the present century the work of a number of mathematicians has created new methods which make possible a very simple treatment of our problem; it is the purpose of the following pages to give an account of these methods which, while as short as possible, shall yet be essentially complete.

CHAPTER I

MÖBIUS TRANSFORMATION

5. Conformal representation in general.

It is known from the theory of functions that an analytic function $w=f(z)$, which is regular and has a non-zero differential coefficient at the point $z=z_0$, gives a continuous one-one representation of a certain neighbourhood of the point z_0 of the z-plane on a neighbourhood of a point w_0 of the w-plane.

Expansion of the function $f(z)$ gives the series

$$\left.\begin{aligned} w-w_0 &= A(z-z_0)+B(z-z_0)^2+\ldots, \\ A &\neq 0; \end{aligned}\right\} \qquad \ldots\ldots(5\cdot1)$$

and if we write

$$z-z_0=re^{it}, \quad A=ae^{i\lambda}, \quad w-w_0=\rho e^{iu}, \qquad \ldots\ldots(5\cdot2)$$

where t, λ, and u are real, and r, a, and ρ are positive, then (5·1) may be written

$$\left.\begin{aligned} \rho e^{iu} &= are^{i(\lambda+t)}\{1+\phi(r,t)\}, \\ &\lim_{r\to0}\phi(r,t)=0. \end{aligned}\right\} \qquad \ldots\ldots(5\cdot3)$$

This relation is equivalent to the following two relations:

$$\left.\begin{aligned} \rho=ar\{1+\alpha(r,t)\}, &\quad u=\lambda+t+\beta(r,t), \\ \lim_{r\to0}\alpha(r,t)=0, &\quad \lim_{r\to0}\beta(r,t)=0. \end{aligned}\right\} \qquad \ldots\ldots(5\cdot4)$$

When $r=0$ the second of these relations becomes

$$u=\lambda+t, \qquad \ldots\ldots(5\cdot5)$$

and expresses the connection between the direction of a curve at the point z_0 and the direction of the corresponding curve at the point w_0. Equation (5·5) shows in particular that the representation furnished by the function $w=f(z)$ at the point z_0 is isogonal. Since the derivative $f'(z)$ has no zeros in a certain neighbourhood of z_0, it follows that the representation effected by $f(z)$ of a neighbourhood of z_0 on a portion of the w-plane is not only continuous but also conformal.

The first of the relations (5·4) can be expressed by saying that "infinitely small" circles of the z-plane are transformed into infinitely small circles of the w-plane. Non-trivial conformal transformations exist however for which this is also true of *finite* circles; these transformations will be investigated first.

6. Möbius Transformation.

Let A, B, C denote three real or complex constants, $\bar{A}$, $\bar{B}$, $\bar{C}$ their conjugates, and x, $\bar{x}$ a complex variable and its conjugate; then the equation

$$(A+\bar{A})\,x\bar{x}+Bx+\bar{B}\bar{x}+C+\bar{C}=0 \qquad \ldots\ldots(6\cdot1)$$

represents a real circle or straight line provided that

$$B\bar{B}>(A+\bar{A})(C+\bar{C}). \qquad \ldots\ldots(6\cdot2)$$

Conversely every real circle and every real straight line can, by suitable choice of the constants, be represented by an equation of the form (6·1) satisfying condition (6·2). If now in (6·1) we make any of the substitutions

$$y=x+\lambda, \qquad \ldots\ldots(6\cdot3)$$

$$y=\mu x, \qquad \ldots\ldots(6\cdot4)$$

or

$$y=\frac{1}{x}, \qquad \ldots\ldots(6\cdot5)$$

the equation obtained can be brought again into the form (6·1), with new constants A, B, C which still satisfy condition (6·2). The substitution (6·5) transforms those circles and straight lines (6·1) for which $C+\bar{C}=0$, i.e. those which pass through the point $x=0$, into straight lines; we shall therefore regard straight lines as circles which pass through the point $x=\infty$.

7. If we perform successively any number of transformations (6·3), (6·4), (6·5), taking each time arbitrary values for the constants λ, μ, the resulting transformation is always of the form

$$y=\frac{\alpha x+\beta}{\gamma x+\delta}; \qquad \ldots\ldots(7\cdot1)$$

here α, β, γ, δ are constants which necessarily satisfy the condition

$$\alpha\delta-\beta\gamma\neq0, \qquad \ldots\ldots(7\cdot2)$$

since otherwise the right-hand member of (7·1) would be either constant or meaningless, and (7·1) would not give a transformation of the x-plane. Conversely, any bilinear transformation (7·1) can easily be obtained by means of transformations (6·3), (6·4), (6·5), and hence (7·1) also transforms circles into circles.

The transformation (7·1) was first studied by *Möbius* (7) (1790–1868), and will therefore be called *Möbius' Transformation.*

8. The transformation inverse to (7·1), namely

$$x=\frac{-\delta y+\beta}{\gamma y-\alpha}, \quad (-\delta)(-\alpha)-\beta\gamma\neq0, \qquad \ldots\ldots(8\cdot1)$$

is also a Möbius' transformation. Further, if we perform first the transformation (7·1) and then a second Möbius' transformation

$$z = \frac{\alpha_1 y + \beta_1}{\gamma_1 y + \delta_1}, \quad \alpha_1 \delta_1 - \beta_1 \gamma_1 \neq 0,$$

the result is a third Möbius' transformation

$$z = \frac{\mathrm{A}x + \mathrm{B}}{\Gamma x + \Delta},$$

with non-vanishing determinant

$$\mathrm{A}\Delta - \mathrm{B}\Gamma = (\alpha\delta - \beta\gamma)(\alpha_1\delta_1 - \beta_1\gamma_1) \neq 0.$$

Thus we have the theorem: *the aggregate of all Möbius' transformations forms a group.*

9. Equations (7·1) and (8·1) show that, if the x-plane is closed by the addition of the point $x = \infty$, *every Möbius' transformation is a one-one transformation of the closed x-plane into itself.* If $\gamma \neq 0$, the point $y = \alpha/\gamma$ corresponds to the point $x = \infty$, and $y = \infty$ to $x = -\delta/\gamma$; but if $\gamma = 0$ the points $x = \infty$ and $y = \infty$ correspond to each other.

From (7·1) we obtain

$$\frac{dy}{dx} = \frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2},$$

so that, by § 5, the representation is conformal except at the points $x = \infty$ and $x = -\delta/\gamma$. In order that these two points may cease to be exceptional we now extend the definition of conformal representation as follows: a function $y = f(x)$ will be said to transform the neighbourhood of a point x_0 conformally into a neighbourhood of $y = \infty$ if the function $\eta = 1/f(x)$ transforms the neighbourhood of x_0 conformally into a neighbourhood of $\eta = 0$; also $y = f(x)$ will be said to transform the neighbourhood of $x = \infty$ conformally into a neighbourhood of y_0 if

$$y = \phi(\xi) = f(1/\xi)$$

transforms the neighbourhood of $\xi = 0$ conformally into a neighbourhood of y_0. In this definition y_0 may have the value ∞.

In virtue of the above extensions we now have the theorem: *every Möbius' transformation gives a one-one conformal representation of the entire closed x-plane on the entire closed y-plane.*

10. Invariance of the cross-ratio.

Let x_1, x_2, x_3, x_4 denote any four points of the x-plane, and y_1, y_2, y_3, y_4 the points which correspond to them by the Möbius' transformation (7·1). If we suppose in the first place that all the numbers x_i, y_i

are finite, we have, for any two of the points,

$$y_k - y_i = \frac{\alpha x_k + \beta}{\gamma x_k + \delta} - \frac{\alpha x_i + \beta}{\gamma x_i + \delta} = \frac{\alpha\delta - \beta\gamma}{(\gamma x_k + \delta)(\gamma x_i + \delta)}(x_k - x_i),$$

and consequently, for all four,

$$\frac{(y_1 - y_4)(y_3 - y_2)}{(y_1 - y_2)(y_3 - y_4)} = \frac{(x_1 - x_4)(x_3 - x_2)}{(x_1 - x_2)(x_3 - x_4)}. \qquad \ldots\ldots(10{\cdot}1)$$

The expression

$$\frac{(x_1 - x_4)(x_3 - x_2)}{(x_1 - x_2)(x_3 - x_4)}$$

is called the cross-ratio of the four points x_1, x_2, x_3, x_4, so that, by (10·1), the cross-ratio is invariant under any Möbius' transformation.

A similar calculation shows that equation (10·1), when suitably modified, is still true if one of the numbers x_i or one of the numbers y_i is infinite; if, for example, $x_2 = \infty$ and $y_1 = \infty$,

$$\frac{y_3 - y_2}{y_3 - y_4} = \frac{x_1 - x_4}{x_3 - x_4}. \qquad \ldots\ldots(10{\cdot}2)$$

11. Let x_1, x_2, x_3 and y_1, y_2, y_3 be two sets each containing three unequal complex numbers. We will suppose in the first place that all six numbers are finite. The equation

$$\frac{(y_1 - y)(y_3 - y_2)}{(y_1 - y_2)(y_3 - y)} = \frac{(x_1 - x)(x_3 - x_2)}{(x_1 - x_2)(x_3 - x)} \qquad \ldots\ldots(11{\cdot}1)$$

when solved for y yields a Möbius' transformation which, as is easily verified, transforms each point x_i into the corresponding point y_i*, and § 10 now shows that it is the *only* Möbius' transformation which does so. This result remains valid when one of the numbers x_i or y_i is infinite, provided of course that equation (11·1) is suitably modified.

12. Since a circle is uniquely determined by three points on its circumference, § 11 may be applied to find Möbius' transformations which transform a given circle into a second given circle or straight line. Thus, for example, by taking $x_1 = 1$, $x_2 = i$, $x_3 = -1$ and $y_1 = 0$, $y_2 = 1$, $y_3 = \infty$, we obtain the transformation

$$y = i\,\frac{1 - x}{1 + x}, \qquad \ldots\ldots(12{\cdot}1)$$

i.e. one of the transformations which represent the circle $|x| = 1$ on the real axis, and the *interior* $|x| < 1$ of the unit-circle on the *upper* half of the y-plane. By a different choice of the six points x_i, y_i we can represent the exterior $|x| > 1$ of the unit-circle on this same half-plane.

* The determinant of this transformation has the value

$$\alpha\delta - \beta\gamma = (y_1 - y_2)(y_1 - y_3)(y_2 - y_3)(x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

In particular by taking the three points y_i on the same circle as the points x_i we can transform the interior of this circle into itself or into the exterior of the circle according as the points x_1, x_2, x_3 and y_1, y_2, y_3 determine the same or opposite senses of description of the perimeter. If, for example, in (11·1) we put $y_1 = 0$, $y_2 = 1$, $y_3 = \infty$, and then successively $x_1 = 1$, $x_2 = \infty$, $x_3 = 0$ and $x_1 = \infty$, $x_2 = 1$, $x_3 = 0$, we obtain the two transformations

$$y = (x-1)/x \quad \text{and} \quad y = 1/x; \qquad \text{......(12·2)}$$

the first transforms the upper half-plane into itself, whereas the second transforms it into the lower half-plane.

13. Pencils of circles.

Since a Möbius' transformation is conformal it transforms orthogonal circles into orthogonal circles. We shall now show that, *given any two circles A and B, we can find a Möbius' transformation which transforms them either into two straight lines or into two concentric circles.*

If A and B have at least one common point P, then any Möbius' transformation whereby P corresponds to the point ∞ transforms A and B into straight lines; these lines intersect or are parallel according as A and B have a common point other than P, or not.

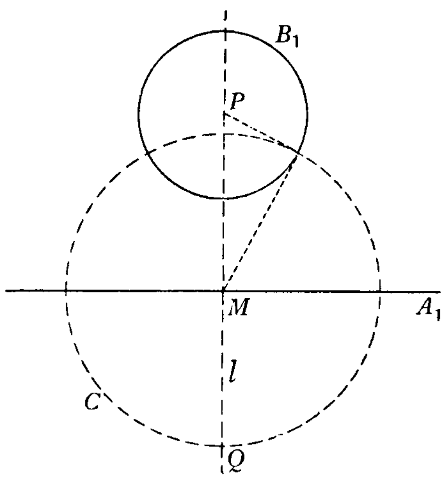

Fig. 1

If A and B have no common point, first transform the circle A by a Möbius' transformation into a straight line A_1, and let B_1 be the circle corresponding to B; A_1 and B_1 do not intersect. Draw the straight line l through the centre of B_1 perpendicular to A_1; let the foot of this perpendicular be M. With centre M draw the circle C cutting B_1 orthogonally. By a second Möbius' transformation we can transform the circle C and the straight line l into two (orthogonal) intersecting straight lines; A_1, B_1 are thereby transformed into two circles A_2, B_2, which cut both these straight lines orthogonally and are therefore *concentric.*

14. Given two intersecting straight lines there is a family of concentric circles orthogonal to both; given two parallel straight lines there is a family of parallel straight lines orthogonal to both; and given two concentric circles there is a family of intersecting straight lines orthogonal to both. Each of these families of circles or straight lines consists of all circles (or straight lines) of the plane which cut both the given lines or circles orthogonally. Since a Möbius' transformation is isogonal it follows that: *given any two circles A, B, there exists exactly one one-parametric family of circles which cut A and B orthogonally; this family is called the pencil of circles conjugate to the pair A, B.*

If the circles A and B intersect in two points P, Q of the plane, no two circles of the conjugate pencil can intersect, and the pencil is then said to be *elliptic.* No circle of the pencil passes through either of the points P, Q, which are called the *limiting points* of the pencil.

Secondly, if A and B touch at a point P, the conjugate pencil consists of circles all of which touch at P, and is called *parabolic*; P is the *common point* (*Knotenpunkt*) of the pencil.

Lastly, if A and B have no point in common, the conjugate pencil consists of all circles which pass through two fixed points, the *common points* of the pencil, and is called *hyperbolic.*

15. Considering the three types of pencils of circles as defined in § 14, we see that if C, D are any two circles of the pencil conjugate to A, B, then A, B belong to the pencil conjugate to C, D. This pencil containing A, B is independent of the choice of the two circles C, D, and we therefore have the following theorem: *there is one and only one pencil of circles which contains two arbitrarily given circles;* i.e. *a pencil of circles is uniquely determined by any two of its members.*

We see further from the three standard forms of pencils that: *through every point of the plane which is neither a limiting point nor a common point of a given pencil of circles there passes exactly one circle of the pencil.*

16. Bundles of circles.

Let A, B, C be three circles which do not all pass through a common point P. If A, B have no common point we can transform them (§ 13) by a Möbius' transformation into concentric circles A_1, B_1, and that common diameter of A_1 and B_1 which cuts C_1 (the circle into which C is transformed) orthogonally is a circle of the plane cutting all three circles A_1, B_1, C_1 orthogonally. Hence a circle exists which cuts all three circles A, B, C orthogonally.

Secondly, if A and B touch, there is a Möbius' transformation which transforms them into two parallel straight lines, and C into a circle C_1. Since C_1 has one diameter perpendicular to the two parallel straight lines, a circle exists in this case also cutting all three circles A, B, C orthogonally.

Finally, if A and B have two points in common, there is a Möbius' transformation which transforms them into two straight lines intersecting at a point O, and C into a circle C_1 which does not pass through O. Two cases must now be distinguished: if O lies *outside* the circle C_1 there is

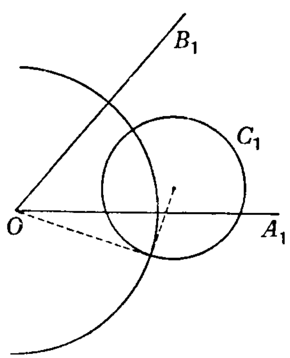

Fig. 2

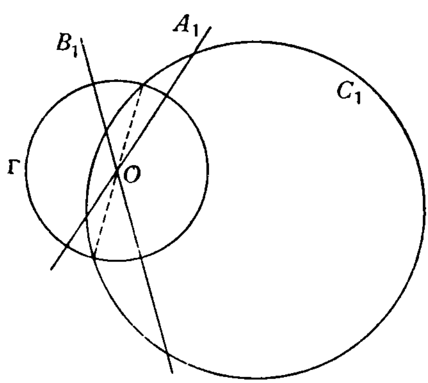

Fig. 3

again a circle cutting A_1, B_1, and C_1 orthogonally; whereas if O lies *inside* C_1 there is a circle Γ such that each of the circles A_1, B_1, C_1 intersects Γ at the extremities of a diameter of Γ.

We have thus proved the following theorem: *any three co-planar circles must satisfy at least one of the following conditions: the three circles have a common orthogonal circle K, or they pass through a common point, or they can be transformed by a Möbius' transformation into three circles which cut a fixed circle Γ at the extremities of a diameter of Γ.* It follows readily from the proof given that if the three circles A, B, C do not belong to the same pencil the circle K is unique; further, it will be proved below that three given circles *cannot satisfy more than one* of the three conditions enumerated.

17. We now define three types of families of circles which we call *bundles of circles.*

An *elliptic bundle of circles* consists of all circles of the plane which cut a fixed circle Γ at the extremities of a diameter of Γ. The circle Γ itself belongs to the bundle and is called the *equator* of the bundle.

A *parabolic bundle of circles* consists of all circles of the plane which pass through a fixed point, the *common point* of the bundle.

A *hyperbolic bundle of circles* consists of all circles of the plane which cut a fixed circle or straight line orthogonally.

These three figures are *essentially* distinct: every pair of circles of an elliptic bundle intersect at two points; every pair of circles of a parabolic bundle either intersect at two points or touch one another; but a hyperbolic bundle contains pairs of circles which have no common point.

18. Bundles of circles nevertheless possess very remarkable common properties. For example: *if A, B are two circles of a bundle, all the circles of the pencil which contains A, B belong to this bundle.* For a parabolic bundle the truth of this theorem is obvious; for a hyperbolic bundle it follows from the fact that the orthogonal circle of the bundle cuts the circles A, B—and therefore cuts every circle of the pencil containing A, B—orthogonally; and for an elliptic bundle it follows from an elementary theorem of Euclid.

The proof of the following theorem is equally simple: *if a plane contains a bundle of circles and an arbitrary point P, which, if the bundle is parabolic, does not coincide with the common point of the bundle, then P lies on an infinite number of circles of the bundle, and these circles through P form a pencil.*

19. Let A, B, C be three circles of a bundle which do not belong to the same pencil, and let D be any fourth circle of the bundle; then, starting with A, B, C we can, by successive construction of pencils, arrive at a pencil of circles which contains D, and all of whose members are circles of the bundle. For there is on D at least one point P which is neither a common point nor a limiting point of either of the two pencils determined by A, B and by A, C and which does not lie on A; we can therefore draw through P two circles E, F, so that E belongs to the pencil A, B, and F to the pencil A, C. The circles E, F are distinct, since A, B, C do not belong to the same pencil, and the second theorem of § 18 now shows that D belongs to the pencil determined by E, F.

It follows that a bundle of circles is uniquely determined by any three of its members which do not belong to the same pencil, and in particular