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978-0-521-09156-5 - The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coordinates

E. A. Maxwell

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