THE METHODS
OF
PLANE PROJECTIVE GEOMETRY
BASED ON THE USE
OF
GENERAL HOMOGENEOUS
COORDINATES

THE METHODS OF PLANE PROJECTIVE GEOMETRY BASED ON THE USE OF GENERAL HOMOGENEOUS COORDINATES

BY

E. A. MAXWELL

Fellow of Queens' College, Cambridge

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DEDICATED

TO MY

FATHER AND MOTHER

ON THE OCCASION OF MY

FATHER'S EIGHTIETH BIRTHDAY

24 AUGUST 1945

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PREFACE

THIS BOOK is an introduction to the methods of projective geometry, based on the use of homogeneous coordinates. It is intended for pupils in their last year at school and their first year at the university. It is written as a study of *methods* and not as a catalogue of theorems; and I hope that a student reading it will have nothing to unlearn as he proceeds to apply these methods to study the geometry of figures in three dimensions or in higher space.

The first three chapters introduce homogeneous coordinates, the equation of the straight line, duality, one-one algebraic correspondence and cross-ratio. The fourth chapter (which some teachers may prefer to leave until a later part of the course) deals with the conic, treated parametrically. In the fifth and sixth chapters, the standard properties of conics are obtained, care being taken to show the importance of an intelligent choice of the triangle of reference. The seventh chapter applies the theory of one-one correspondence to the study of conics, including Chasles's theorem. The eighth chapter gives an account of the quadrilateral and the quadrangle, and of pencils of conics through four points or touching four lines; in the ninth chapter more general pencils are considered. The methods given earlier in the book are applied in the tenth chapter to the study of various classical properties; these properties are not discussed in much detail, but it is hoped that the reader will be encouraged to read more advanced works.

In the first ten chapters the ideas of length and angle are not used at all. The eleventh chapter gives the rules for interpreting the projective results metrically, and in the twelfth chapter these rules are applied to a variety of problems. Here, again, there is no attempt to be exhaustive; the methods are the important things.

Though the book is mainly on the use of homogeneous coordinates, I have not hesitated to introduce the methods of Pure Geometry where they seemed most suited to my immediate purpose. The good geometer should move freely in both Pure and Analytical Geometry, and he will find here the use of both.

A word should be said about the examples. I have searched the

Cambridge Examination Papers for many years back, and I have also used other sources when I could get a copy conveniently. I found the arrangement hard; the trouble with examples in geometry is that they can be tackled by many different methods, as any examiner knows. I have tried to place them where they arise most naturally, and I suggest that, as the reader progresses through the book, he should turn back to the examples in earlier chapters to see whether he can solve them more simply in the light of his increased experience. I have added some routine examples in the first few chapters till the reader gets the 'feel' of the subject, but later examples are almost entirely taken from papers; an exception had to be made in Chapter IV, where the topic seems to have eluded the examiner. There are also no examples to Chapter XI, as it is really an introduction to the following chapter.

The initials at the end of the examples have the following meanings:

F. Ferguson Scholarship Examinations;

G. Goldsmiths' Company's Exhibitions;

L. University of London Honours Examinations;

M.T. I or II. Mathematical Tripos, Parts I or II;

O. and C. Oxford and Cambridge Schools Examination Board;

P. Preliminary Examination in the University of Cambridge;

C.S. Cambridge Scholarship Examinations;

W. Warwickshire Scholarship Examinations.

I am indebted to these bodies for permission to use the examples.

No attempt is made to record the sources of the results. My interest in the subject was aroused by the books of Prof. H. F. Baker and the lectures of Mr F. P. White, both of St John's College, Cambridge. I have also consulted several text-books, but should not like to say exactly where I met some of the methods. I do not think that any earlier text-book has been written from exactly the point of view adopted here.

I have been extremely fortunate in the help received in the preparation of this book. The original manuscript was read first by an analyst and algebraist, Mr W. L. Ferrar, Fellow and Bursar of Hertford College, Oxford, and then by a schoolmaster who 'doesn't profess to be a geometer', Mr G. L. Parsons, of Merchant Taylors' School, Secretary of the Mathematical Association. Those who know these gentlemen will appreciate that I went to them expecting candid criticism, and I was not disappointed. I am most grateful to them for the considerable trouble which they took in removing errors and obscurities. Later, when the manuscript was submitted to the Cambridge University Press, it was read on their behalf by a distinguished geometer, whose comments helped me to remove further obscurities, and then by another who gave the book general approval. I am indebted to both of them for their help and advice. Finally, Dr J. A. Todd, Lecturer in the University of Cambridge, read the proofs and verified all the examples. The reader, following in his footsteps, will understand the work involved, and I am deeply grateful to him for much trouble taken and for the soundness of his advice.

To the staff of the Cambridge University Press I would express my thanks for their care in printing and courtesy in helping me when lapses in manuscript called for alterations.

E. A. M.

QUEENS' COLLEGE,
CAMBRIDGE.
April 1946

A few small corrections have been made on re-printing.

E. A. M.

December 1947.

INTRODUCTION

THIS BOOK contains an elementary account of Projective Geometry, based mainly upon the use of homogeneous coordinates. No knowledge of geometry is assumed explicitly, but the reader will almost certainly have some experience of cartesian analytical geometry, probably including conic sections. In this introduction we record a few general principles which will be used in the course of the work.

1. Determinants. We assume that the reader has some idea of the elementary properties of determinants, including expansions and minors. In particular, we shall assume the following two converse properties, stated for three variables, though the result is perfectly general:

(i) If there are values of x, y, z, not all zero, such that the three equations

$$a_1x+b_1y+c_1z=0,\quad a_2x+b_2y+c_2z=0,\quad a_3x+b_3y+c_3z=0$$

hold simultaneously, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(ii) If

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

then there exist numbers x, y, z, not all zero, such that the three equations

$$a_1x+b_1y+c_1z=0,\quad a_2x+b_2y+c_2z=0,\quad a_3x+b_3y+c_3z=0$$

hold simultaneously.

If Δ denotes the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the *minors* of Δ, with their correct signs attached, are given by the relations

$$A_1 \equiv b_2c_3 - b_3c_2, \quad B_1 \equiv c_2a_3 - c_3a_2, \quad C_1 \equiv a_2b_3 - a_3b_2;$$
$$A_2 \equiv b_3c_1 - b_1c_3, \quad B_2 \equiv c_3a_1 - c_1a_3, \quad C_2 \equiv a_3b_1 - a_1b_3; \text{ etc.}$$

The value of the expression

$$a_iA_j + b_iB_j + c_iC_j \quad (i, j = 1, 2, \text{ or } 3)$$

is equal to Δ when i, j are equal, and to zero when i, j are unequal.

The determinant
$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$
whose elements are the minors of Δ, is equal in value to Δ^2. Each *minor* of this determinant is equal to Δ times the corresponding *element* of the determinant Δ. Thus

$$B_2C_3 - B_3C_2 = a_1\Delta.$$

2. Infinite values. We shall frequently be concerned with the *ratios* of numbers. Let us consider the ratio a/b. It can take any given value, and that, indeed, in many ways, in the form $\lambda a/\lambda b$, where λ is arbitrary; in particular, it is zero when a is zero, assuming that b does not vanish too. Suppose now that a has some given value, while b becomes smaller and smaller; the ratio a/b becomes larger and larger, and can be made greater than any number we care to name. It is in this sense that we say that the ratio a/b is infinite; in fact, the two statements 'a/b is infinite' and 'b/a is zero' are equivalent.

Logically, we ought throughout this work to consider numbers as ratios in the above sense. But there are times when the notation becomes cumbersome and fails to appeal to the eye (an important requirement of mathematical notation). For example, we shall have to discuss the ratios

$$\theta^2 : \theta\theta' : \theta'^2,$$

which we might express in either of the alternative forms

$$(\theta/\theta')^2 : (\theta/\theta') : 1 \quad \text{or} \quad 1 : (\theta'/\theta) : (\theta'/\theta)^2.$$

The first form shows that the ratios include the set

$$0:0:1 \quad \text{when} \quad \theta=0,$$

and the second form shows that they include the set

$$1:0:0 \quad \text{when} \quad \theta'=0.$$

We shall, however, often find it convenient to regard the ratios as given by the single number θ, and to express them in the form

$$\theta^2:\theta:1,$$

in which the ratios

$$0:0:1 \quad \text{arise when } \theta \text{ is zero,}$$

and the ratios

$$1:0:0 \quad \text{arise when } \theta \text{ is infinite.}$$

The infinite value of θ, in fact, corresponds to the zero value of θ'.

We shall assume that a, b do not vanish simultaneously, because each ratio a/b or b/a would then be indeterminate.

It will sometimes be convenient to use the symbol ∞ to denote an infinite value.

3. Reality. We assumed implicitly in the preceding paragraph that the numbers considered were all real; such an assumption was indeed necessary to give that introduction to the idea of infinity its meaning. We shall, however, in subsequent work take the symbols $a, b, \theta, x, \ldots$ to be *complex* (that is, of the form $A+iB$, where A, B are real and $i=\sqrt{(-1)}$) unless the contrary is stated. For economy of language, we shall then use the phrase '*θ is infinite*' to mean '*θ can be expressed in the form θ_1/θ_2, where θ_2 is zero*', whether θ is real or complex.

As our aim is to keep this exposition fairly simple, we shall not refer much to the reality of the numbers used, until the last two chapters; but the reader should always have in mind that complex numbers are intended. A short sketch of some of the implications is given in Chapter I, § 3.