CHAPTER I

HOMOGENEOUS COORDINATES AND THE STRAIGHT LINE

1. Coordinates. The position of a point in a plane may be defined by means of *two* numbers. For example, if x, y are the distances of a point P from two given perpendicular lines, then (subject to the conventions on sign given in any book on cartesian coordinates) the position of P is uniquely determined when x, y are known; and, conversely, if the position of P is assigned, then x, y are determined. The values of x, y are known as the *cartesian coordinates* of P.

It is, however, possible to define the position of P in many other ways. As an example, let us consider the case of *areal coordinates*. Suppose that we are given a triangle XYZ, and that P is a point which (to avoid irrelevant complications) we take inside the triangle. It is easy to see that the position of P is determined if we know the areas of the triangles PYZ, PZX, PXY, and conversely. If we denote these areas by Δ_1, Δ_2, Δ_3 and the area of the triangle XYZ by Δ, then the areal coordinates of P are defined as x, y, z, where

$$x = \Delta_1/\Delta, \quad y = \Delta_2/\Delta, \quad z = \Delta_3/\Delta.$$

Here it will be noticed that we have *three* coordinates, but there is an identical relation connecting them, namely

$$x+y+z = 1.$$

In like manner, we could, if we had wished, have expressed the cartesian coordinates of P in terms of three numbers x, y, z, taking the distances as

$$x/z, \quad y/z.$$

In this case, we should have had the identical relation

$$z = 1.$$

We shall see later that this remark is not as trivial as might appear at first sight.

2. Importance of ratios. From these and other examples we see that, while the position of a point P can be determined by *two* coordinates, it is often convenient to use *three*, which must then be connected by some relation. We shall not find it necessary to specify this relation explicitly; in other words, we shall not find it necessary to specify any particular system of coordinates. Our work, in fact, will be independent of metrical ideas, and such notions as length, area, angle will not be required. It will be found that this general conception, while not increasing the difficulty, gives a more unified view of the subject. We shall, however, conclude the book by showing how the general results can be interpreted if required in terms of ordinary metrical geometry.

Before we proceed to define our more abstract geometry, we note two details:

(i) The relation between the coordinates x, y, z takes (for ordinary coordinate systems, at any rate) the form

$$ax+by+cz = 1.$$

Our examples were

Cartesian coordinates: $a = b = 0$, $c = 1$,

Areal coordinates: $a = b = c = 1$.

(ii) We shall find in our work that the properties of figures are expressed by *algebraic equations* among the variables x, y, z. By using the above linear relation, we can ensure that these equations are *homogeneous*. For example, we could use the relation $z = 1$, if it were the relevant one, to express the equations

$$x^2+y^2+2gx+2fy+c = 0, \quad y^2 = 4ax$$

in the forms

$$x^2+y^2+2gxz+2fyz+cz^2 = 0, \quad y^2 = 4axz$$

respectively.

It follows that our work will involve the study of the *ratios* $x : y : z$, of which *two* are independent. We therefore return to the idea of two numbers from which we began. Which two ratios we choose is immaterial, and will usually be undefined; we might, for example, consider x/z and y/z, or x/y and z/y, according to convenience. The point to remember is that the coordinates x, y, z

involve two independent ratios, and that the equations in which they occur are homogeneous algebraic equations.

From now on, until we come to consider metrical properties in the last two chapters, we shall not require to specify the identical relation.

3. The homogeneous coordinates; the complex projective plane. We make the following basic assumption:

The position of a point in a plane can be uniquely defined by the ratios of three coordinates x, y, z, and, conversely, these ratios define a point of the plane uniquely.

Different systems of coordinates may be chosen, and the corresponding ratios for a definite point of the plane will vary from system to system. We have already given examples of two such systems, so our basic assumption is founded on reasonable experience.

The assumption implies that, in a given system of coordinates, two sets of coordinates with the same ratios, such as $(1, 2, 3)$ and $(-3, -6, -9)$, represent the same point of the plane. The three coordinates determine two *independent* ratios.

If one of the coordinates is zero, as in $(0, 1, 2)$, we can take the two ratios as 0 and $\frac{1}{2}$. If two of the coordinates are zero, as in $(0, 0, 2)$, we can take the two ratios as 0 and 0 (i.e. x/z and y/z). *We assume that x, y, z are not all zero simultaneously.*

In speaking of the point with coordinates x, y, z we shall often call it simply 'the point (x, y, z)'. If we have given the point a name, say P, we shall speak of it as '$P(x, y, z)$'.

Note. In adopting this definition of homogeneous coordinates, we have implicitly extended the Euclidean conception of a plane. Without going into great detail, we can show how this has happened by considering ordinary cartesian coordinates, usually denoted by x, y, and expressing them in the homogeneous form x'/z', y'/z'; there are then three homogeneous coordinates x', y', z', supposed real for the moment. When z' is not zero, we can regard the coordinates as defining the points in an ordinary Euclidean plane. But if in addition we allow z' to take the value zero (x', y' not being zero) we must superpose, as it were, to the Euclidean plane a system of points, namely, a system whose coordinates are 'infinite' in the sense of

Introduction, § 2. If, finally, we allow x', y', z' to take complex values too, then we must consider the plane to be further augmented by a corresponding system of points. The plane so 'covered' by the whole system of points arising from all possible sets of values of x', y', z' (not all zero) is called the *complex projective plane*.

In the early stages of his reading, the student should not bother too much about these problems. When he comes to the two last chapters, however, he will find it necessary to consider them more deeply, and a brief note is added there for his guidance.

4. The straight line. Euclid's definition of a straight line is not readily adaptable to our needs, and we therefore seek an alternative definition. In order to agree with our intuitive ideas, the new definition must satisfy the following conditions:

(i) a straight line is determined by *any* two of its points;

(ii) two straight lines have one common point.

Suppose, then, that $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are two given points. *We define the line AB to consist of the points $P(x, y, z)$ for which a value of the ratio λ/μ can be found such that*

$$x = \lambda x_1 + \mu x_2, \quad y = \lambda y_1 + \mu y_2, \quad z = \lambda z_1 + \mu z_2.$$

Every value of λ/μ (or μ/λ) determines one and only one point, which is called *a point on the line AB*. In particular, $\lambda = 0$ determines B and $\mu = 0$ determines A. The two values λ, μ cannot vanish simultaneously.

The coordinates of P satisfy the relation, found by eliminating the ratios $-1 : \lambda : \mu$ as in Introduction, § 1,

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix} = 0.$$

The relation becomes, on expansion,

$$(y_1 z_2 - y_2 z_1) x + (z_1 x_2 - z_2 x_1) y + (x_1 y_2 - x_2 y_1) z = 0,$$

and this equation, being of the form

$$lx + my + nz = 0,$$

is homogeneous and of degree one. Such an equation is called a *linear* equation.

Conversely, *every point* $P(x, y, z)$ *whose coordinates satisfy the relation*

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix} = 0$$

does lie on the line AB. For, by Introduction, § 1, there then exist numbers p, q, r such that

$$px + qx_1 + rx_2 = 0, \text{ etc.,}$$

and p cannot be zero, otherwise the points A, B would not be distinct. Dividing by p and writing $q/p = -\lambda$, $r/p = -\mu$, we find the relations

$$x = \lambda x_1 + \mu x_2, \quad y = \lambda y_1 + \mu y_2, \quad z = \lambda z_1 + \mu z_2,$$

which show, by definition, that P lies on the line AB.

It is customary to interchange rows and columns and to write the *equation of the line* AB, as it is called, in the form

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

An immediate corollary of the preceding work is that, *if the points* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *are collinear, then*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Conversely, *if this determinant vanishes, then the points are collinear.*

5. Properties of the straight line.

(i) *A straight line is determined by* ANY *two of its points.* Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be two given points, as in § 4. The points of the line AB are those whose coordinates satisfy relations of the type

$$x = \lambda x_1 + \mu x_2, \text{ etc.}$$

Now suppose that C, D are two given points of the line AB, where the coordinates of C are

$$px_1 + qx_2, \quad py_1 + qy_2, \quad pz_1 + qz_2,$$

and the coordinates of D are

$$p'x_1 + q'x_2, \quad p'y_1 + q'y_2, \quad p'z_1 + q'z_2.$$

By definition, the point $P(x, y, z)$ belongs to the line CD if values of λ, μ exist such that

$$x = \lambda(px_1+qx_2)+\mu(p'x_1+q'x_2),$$

with similar results for y, z. Rearranging, we have

$$x = (\lambda p+\mu p')\,x_1+(\lambda q+\mu q')\,x_2,$$
$$y = (\lambda p+\mu p')\,y_1+(\lambda q+\mu q')\,y_2,$$
$$z = (\lambda p+\mu p')\,z_1+(\lambda q+\mu q')\,z_2,$$

and the point P therefore lies also on the line AB, as required.

(ii) *Every linear equation does determine a line, and determine it uniquely.* Consider the equation

$$lx+my+nz = 0.$$

By inspection, the coordinates of the points $Q(-n, 0, l)$, $R(m, -l, 0)$ satisfy the equation. Now the equation of the straight line QR is

$$\begin{vmatrix} x & y & z \\ -n & 0 & l \\ m & -l & 0 \end{vmatrix} = 0$$

or $$l(lx+my+nz) = 0.$$

The factor l is irrelevant, for it would have been m if we had used the points $R(m, -l, 0)$, $P(0, n, -m)$ and n if we had used the points $P(0, n, -m)$, $Q(-n, 0, l)$; and l, m, n are not all zero. The equation of the line QR is therefore

$$lx+my+nz = 0,$$

and so this equation does determine the points of a line.

Further, the line determined by the above linear equation is *unique* since the two points $Q(-n, 0, l)$, $R(m, -l, 0)$ are the only points on it whose y, z coordinates respectively are zero.

Finally, if we are given the *ratios* $l:m:n$, then they determine a line, namely, the line joining the points $Q(-n, 0, l)$, $R(m, -l, 0)$.

(iii) *Two straight lines have one common point.* Let the equations of the two lines be

$$l_1x+m_1y+n_1z = 0, \quad l_2x+m_2y+n_2z = 0.$$

Solving these equations, we obtain the ratios

$$\frac{x}{m_1n_2-m_2n_1} = \frac{y}{n_1l_2-n_2l_1} = \frac{z}{l_1m_2-l_2m_1};$$

hence the ratios of x, y, z are the ratios of

$$m_1n_2-m_2n_1, \quad n_1l_2-n_2l_1, \quad l_1m_2-l_2m_1.$$

These ratios determine a unique point unless, exceptionally, they all vanish. The point so determined lies on each of the lines.

In the exceptional case when the ratios all vanish, we have

$$\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2},$$

so that the two given equations represent the same line.

(iv) *The condition that the three straight lines*

$$l_1x+m_1y+n_1z=0, \quad l_2x+m_2y+n_2z=0, \quad l_3x+m_3y+n_3z=0$$

should have a common point is that

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0.$$

For this is the condition (Introduction, § 1) that there should be a value of the ratios $x:y:z$ satisfying the three given equations.

Conversely, if the determinant vanishes, then there is a value of the ratios $x:y:z$ satisfying the given equations, and so the three lines which they represent have a common point.

Three lines with a common point are said to be *concurrent*.

DEFINITION. The figure formed by three non-concurrent lines is called a *triangle*. The lines are called the *sides* of the triangle, and the three points where two sides meet are called the *vertices*.

6. The triangle of reference. The points $X(1, 0, 0)$, $Y(0, 1, 0)$, $Z(0, 0, 1)$ form a triangle called the *triangle of reference*. The side YZ of the triangle is

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=0,$$

or $x=0$. The sides ZX, XY are similarly $y=0$, $z=0$ respectively.

The reader should verify the following results:

(i) The equation of any straight line through X is

$$my+nz=0.$$

(ii) The coordinates of any point of YZ can be expressed as $(0, y_1, z_1)$. It is sometimes convenient to use the form $(0, 1, \zeta)$, in which the points Y, Z are given by the values zero and infinity respectively of the number ζ.

(iii) If $P(x_1, y_1, z_1)$ is any point of the plane, then the equation of XP is

$$\frac{y}{y_1} = \frac{z}{z_1},$$

and the line XP meets YZ in the point $(0, y_1, z_1)$.

(iv) If $lx+my+nz = 0$ is any line in the plane, then it meets YZ in the point $L(0, -n, m)$, and the equation of LX is

$$my+nz = 0.$$

7. The unit point. If the triangle of reference is given, we can take a system of coordinates in which any assigned point U, not on a side of the triangle of reference, has coordinates $(1, 1, 1)$ as follows:

Suppose that, in any given coordinate system x, y, z the coordinates of U are (α, β, γ), and effect a *transformation* from the system x, y, z to a system x', y', z' by means of the relations

$$x' = x/\alpha, \quad y' = y/\beta, \quad z' = z/\gamma.$$

Then (i) the coordinates of every point of the plane are determined in terms of x', y', z'; (ii) the point (α, β, γ) becomes the point $(1, 1, 1)$; (iii) the triangle of reference is unchanged, since the lines $x=0$, $y=0$, $z=0$ become the lines $x'=0$, $y'=0$, $z'=0$ respectively. We have therefore found a system of coordinates in which U is the point $(1, 1, 1)$. This point is called the *unit point* for that system of coordinates.

We verify that *the points of a line* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *as defined in* §4 *are also the points of the line* AB *when it is so defined in terms of the new coordinates*. In terms of these new coordinates, A, B are the points (x'_1, y'_1, z'_1), (x'_2, y'_2, z'_2), where

$$x'_1 = x_1/\alpha, \quad y'_1 = y_1/\beta, \quad z'_1 = z_1/\gamma;$$

$$x'_2 = x_2/\alpha, \quad y'_2 = y_2/\beta, \quad z'_2 = z_2/\gamma.$$

Also the coordinates of the point $P(\lambda x_1+\mu x_2, \lambda y_1+\mu y_2, \lambda z_1+\mu z_2)$ become, in the new coordinates,

$$\frac{\lambda x_1+\mu x_2}{\alpha}, \quad \frac{\lambda y_1+\mu y_2}{\beta}, \quad \frac{\lambda z_1+\mu z_2}{\gamma},$$

or $$\lambda x_1'+\mu x_2', \quad \lambda y_1'+\mu y_2', \quad \lambda z_1'+\mu z_2',$$

and these are the coordinates of a point of the line $A(x_1', y_1', z_1')$, $B(x_2', y_2', z_2')$, as required.

8. The unit line. We may similarly simplify the equation of the line $lx+my+nz=0$ to the form

$$x+y+z=0$$

(the *unit line*) by means of the relations

$$x'=lx, \quad y'=my, \quad z'=nz.$$

Note that *the two simplifications, unit point and unit line, cannot be effected simultaneously for an arbitrary point as well as for an arbitrary line.* We shall see later (Illustration 2) that the unit line is determined geometrically when the unit point and the triangle of reference are given.

ILLUSTRATION 1. *Theorem of Desargues. If two triangles are in perspective, then the points of intersection of corresponding sides are collinear.*

Two triangles are said to be *in perspective* if the lines joining corresponding vertices are concurrent. Take one of the triangles as triangle of reference XYZ, and let $P(\alpha, \beta, \gamma)$ be the point of intersection of the lines XX', YY', ZZ', where $X'Y'Z'$ is the other triangle. Since X', Y', Z' lie on PX, PY, PZ, we can take their coordinates as

$$X'(\alpha+\lambda, \beta, \gamma), \quad Y'(\alpha, \beta+\mu, \gamma), \quad Z'(\alpha, \beta, \gamma+\nu),$$

on using the fundamental definition for the points of a straight line. (It is assumed that the two triangles do not have any common vertex.)

The equation of the line $Y'Z'$ is

$$\begin{vmatrix} x & y & z \\ \alpha & \beta+\mu & \gamma \\ \alpha & \beta & \gamma+\nu \end{vmatrix} = 0,$$

which meets YZ ($x=0$) where, after a little simplification,

$$\frac{y}{\mu}+\frac{z}{\nu}=0.$$

The point is therefore $(0, \mu, -\nu)$. The other two points are similarly $(-\lambda, 0, \nu)$, $(\lambda, -\mu, 0)$. These three points lie on the straight line

$$\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}=0.$$

The reader should prove the converse result that, if the points of intersection of corresponding sides of two triangles are collinear, then the lines joining corresponding vertices are concurrent.

The point P is called the *centre of perspective* of the two triangles, and the line on which corresponding sides meet is called the *axis of perspective*.

ILLUSTRATION 2. *The polar line of a point with respect to a triangle. Let XYZ be a given triangle and P an arbitrary point. Let PX, PY, PZ meet YZ, ZX, XY respectively in F, G, H and let GH, HF, FG meet YZ, ZX, XY respectively in L, M, N. Then LMN is a straight line, called the* POLAR LINE *of P with respect to the triangle XYZ.*

Let P be (α, β, γ). Then F, G, H are $(0, \beta, \gamma)$, $(\alpha, 0, \gamma)$, $(\alpha, \beta, 0)$, and the equation of GH is

$$-x\beta\gamma+y\gamma\alpha+z\alpha\beta=0.$$

This line meets YZ in the point L for which

$$y\gamma\alpha+z\alpha\beta=0,$$

so that L is the point $(0, \beta, -\gamma)$. The points M, N are similarly $(-\alpha, 0, \gamma)$, $(\alpha, -\beta, 0)$, so that the three points L, M, N all lie on the line

$$x\beta\gamma+y\gamma\alpha+z\alpha\beta=0$$

or

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0.$$

Note that the polar line of the unit point $(1, 1, 1)$ is the unit line $x+y+z=0$, which establishes the geometrical relation referred to in § 8.