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ELEMENTARY MATRICES

AND SOME APPLICATIONS TO DYNAMICS AND DIFFERENTIAL EQUATIONS

by

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ADDENDA ET CORRIGENDA

Additional Definitions.

The *trace* of a square matrix is the sum of the elements in the principal diagonal. It is equal to the sum of the latent roots.

Characteristic vector or *proper vector* is equivalent to "modal column" or "modal row", though less explicit. The term "eigenvector" is sometimes used but is to be strongly deprecated.

p. 33. *Special types of square matrix.* Add

"Matrix is *unitary* when $u^{-1} = \bar{u}'$.

A real unitary matrix is orthogonal but an orthogonal matrix need not be real.

All the latent roots of a unitary matrix have unit modulus.

Matrix is *persymmetric* when the value of u_{ij} depends only on $(i+j)$."

p. 110. Insert dagger † against footnote.

p. 120, 2nd table, 2nd row, for "$r_6 - 2r_9$" read "$r_6 - 3r_9$".

p. 121, line 1. Delete "method of".

p. 121, § 4 12, lines 2, 3. For "the first or the third" read "any".

p. 144, para. beginning at line 8 should read

"If there are p distinct dominant roots $\lambda_1 \lambda_2 \ldots \lambda_p$ and if $\kappa_1 \kappa_2 \ldots \kappa_p$ are the corresponding modal rows, the procedure is as follows. Partition the $(\bar{p}, n)$ matrix $\{\kappa_1, \kappa_2, \ldots, \kappa_p\}$ in the form $[\alpha, \beta]$, where α is a (p, p) submatrix, assumed to be non-singular (rearrangement of the rows of u and columns of $[\alpha, \beta]$ may be necessary to satisfy this condition). In this case the required matrix w is constructed in the partitioned form

$$w = \begin{bmatrix} I, & \alpha^{-1}\beta \\ 0, & 0 \end{bmatrix}$$

and then
$$v = u(I - w) = u\begin{bmatrix} 0, & -\alpha^{-1}\beta \\ 0, & I \end{bmatrix}.$$

Evidently v has p zero columns and hence p zero latent roots. If rearrangement has been required, u must be in the corresponding rearranged form.

The choice of a non-singular submatrix α is a generalization of the choice of a non-zero element κ_{r1} in the elimination of a single dominant root.

This process is in effect that which is applied in the numerical example on p. 330."

p. 150, § 4·21, line 9. For "a machine could no doubt be" read "machines have been" and in line 10 delete "most of".

p. 152, line 5. For "changed" read "reversed".

p. 176, equation (4), denominator of third fraction, for

"$\Delta^{(1)}(\lambda_r)(\lambda - \lambda_r)$" read "$\lambda_r \Delta^{(1)}(\lambda_r)(\lambda - \lambda_r)$".

p. 195, § 6·5, line 7. For "intial" read "initial".

p. 252, equation at bottom, interchange first and third matrices on the right-hand side.

p. 277. The symbol α stands for a set of parameters *and* for the components of the total acceleration of P. One of these should be represented by β, say.

p. 291, § 9·9. The following is a simple alternative proof of the reality of the roots of the determinantal equation $\Delta_m(z) = 0$ when A and E are real and symmetrical.

Let z, k respectively denote any root and its associated modal column, and let $\bar{z}$, $\bar{k}$ be the corresponding conjugates (see § 1·17). Then

$$zAk = Ek. \qquad \ldots\ldots(1)$$

Premultiplication by $\bar{k}'$ yields

$$z\bar{k}'Ak = \bar{k}'Ek, \qquad \ldots\ldots(2)$$

and by transposition

$$zk'A\bar{k} = k'E\bar{k}.$$

The conjugate relation is

$$\bar{z}\bar{k}'Ak = \bar{k}'Ek. \qquad \ldots\ldots(3)$$

Comparison of (2) and (3) gives $z = \bar{z}$, which shows that z is real. Thus by (1) k is real, and by (2) z is positive when the potential energy function is positive and definite.

p. 296, equation (7). An alternative is $Q = k_0'Eq$.

p. 309, equation in (*b*). We may replace $k_r'A$ by $k_r'E$ which may be simpler.

p. 310, § 10·2 (*e*), second sentence should read "The principle shows that first order errors in the mode yield only second order errors in the frequency as calculated by the equation of energy".

Also line 10 should read "used, and when U happens to be symmetrical, a convenient...".

p. 315, line 9 from bottom, for "Rayleigh's principle will next be applied" read "Since U is symmetrical, the extension of Rayleigh's principle given in § 10·2 (*e*) can be applied...".

p. 363, § 12·3, line 4, for "given" read "are given"

p. 396, line above the diagram. For "0·84 degree" read "84 degrees per lb.ft."

PREFACE

The number of published treatises on matrices is not large, and so far as we are aware this is the first which develops the subject with special reference to its applications to differential equations and classical mechanics. The book is written primarily for students of applied mathematics who have no previous knowledge of matrices, and we hope that it will help to bring about a wider appreciation of the conciseness and power of matrices and of their convenience in computation. The general scope of the book is elementary, but occasional discussions of advanced questions are not avoided. The sections containing these discussions, which may with advantage be omitted at the first reading, are distinguished by an asterisk.

The first four chapters give an account of those properties of matrices which are required later for the applications. Chapters I to III introduce the general theory of matrices, while Chapter IV is devoted to various numerical processes, such as the reciprocation of matrices, the solution of algebraic equations, and the calculation of latent roots of matrices by iterative methods.

The remainder of the book is concerned with applications. Chapters V and VI deal in some detail with systems of linear ordinary differential equations with constant coefficients, and Chapter VII contains examples of numerical solutions of systems of linear differential equations with variable coefficients. The last six chapters take up the subject of mechanics. They include an account of the kinematics and dynamics of systems, a separate discussion of motions governed by linear differential equations, illustrations of iterative methods of numerical solution, and a treatment of simple dynamical systems involving solid friction. The part played by friction in the motions of dynamical systems is as yet very incompletely understood, and we have considered it useful to include a very brief description of some experimental tests of the theory.

A considerable number of worked numerical examples has been included. It is our experience that the practical mathematician, whose requirements we have mainly considered, is often able to grasp the significance of a general algebraic theorem more thoroughly when it is illustrated in terms of actual numbers. For examples of applications of dynamical theory we have usually chosen problems relating to the oscillations of aeroplanes or aeroplane structures. Such problems conveniently illustrate the properties of dissipative dynamical systems, and they have a considerable practical importance.

A word of explanation is necessary in regard to the scheme of numbering adopted for paragraphs, equations, tables, and diagrams. The fourth paragraph of Chapter I, for example, is denoted by § 1·4. The two equations introduced in § 1·4 are numbered (1) and (2), but when it is necessary in later paragraphs to refer back to these equations they are described, respectively, as equations (1·4·1) and (1·4·2). Tables and diagrams are numbered in each paragraph in serial order: thus, the two consecutive tables which appear in § 7·13 are called Tables 7·13·1 and 7·13·2, while the single diagram introduced is Fig. 7·13·1.

The list of references makes no pretence to be complete, and in the case of theorems which are now so well established as to be almost classical, historical notices are not attempted. We believe that much of the subject-matter—particularly that relating to the applications—presents new features and has not appeared before in text-books. However, in a field so extensive and so widely explored as the theory of matrices, it would be rash to claim complete novelty for any particular theorem or method.

The parts of the book dealing with applications are based very largely on various mathematical investigations carried out by us during the last seven years for the Aeronautical Research Committee. We wish to express our great indebtedness to that Committee and to the Executive Committee of the National Physical Laboratory for permission to refer to, and expand, a number of unpublished reports, and for granting many other facilities in the preparation of the book. We wish also to record our appreciation of the care which the Staff of the Cambridge University Press has devoted to the printing.

Our thanks are also due to Miss Sylvia W. Skan of the Aerodynamics Department of the National Physical Laboratory for considerable assistance in computation and in the reading of proofs.

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