CHAPTER I

FUNDAMENTAL DEFINITIONS AND ELEMENTARY PROPERTIES

1·1. Preliminary Remarks. Matrices are sets of numbers or other elements which are arranged in rows and columns as in a double entry table and which obey certain rules of addition and multiplication. These rules will be explained in §§ 1·3, 1·4.

Rectangular arrays of numbers are of course very familiar in geometry and physics. For example, an ordinary three-dimensional vector is represented by three numbers called its components arranged in one row, while the state of stress at a point in a medium can be represented by nine numbers arranged in three rows and three columns. However, two points must be emphasised in relation to matrices. Firstly, the idea of a matrix implies the treatment of its elements taken as a whole and in their proper arrangement. Secondly, matrices are something more than the mere arrays of their elements, in view of the rules for their addition and multiplication.

1·2. Notation and Principal Types of Matrix. (*a*) *Rectangular Matrices.* The usual method of representing a matrix is to enclose the array of its elements within brackets, and in general square brackets are used for this purpose.* For instance, the matrix formed from the array

$$\begin{matrix} 1 & 12 & 0 \\ 5 & 6 & 1 \end{matrix}$$

is represented by

$$\begin{bmatrix} 1 & 12 & 0 \\ 5 & 6 & 1 \end{bmatrix}.$$

The meaning of other special brackets will be explained later. If a matrix contains lengthy numbers or complicated algebraic expressions, the elements in the rows can be shown separated by commas to avoid confusion.

The typical element of a matrix such as

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

* Some writers employ thick round brackets or double lines.

can be denoted by A_{ij}, where the suffices i and j are understood to range from 1 to m and from 1 to n, respectively. A convenient abbreviated notation for the complete matrix is then $[A_{ij}]$, but in cases where no confusion can arise it is preferable to omit the matrix brackets and the suffices altogether and to write the matrix simply as A.

The letters i, j are generally used in the sense just explained as suffices for a typical element of a matrix. Specific elements will generally have other suffices, such as m, n, r, s.

(b) *Order*. A matrix having m rows and n columns is said to be of order m by n. For greater brevity, such a matrix will usually be referred to as an $(\overline{m}, n)$ matrix; the bar shows which of the two numbers m, n relates to the rows.*

(c) *Column Matrices and Row Matrices*. A matrix having m elements arranged in a single column—namely, an $(\overline{m}, 1)$ matrix—will be called a *column matrix*. A column of numbers occupies much vertical space, and it is often preferable to adopt the convention that a single row of elements enclosed within braces represents a column matrix. For instance,

$$\{x_1, x_2, x_3\} \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A literal matrix such as the above can be written in the abbreviated form $\{x_i\}$.

In the same way a matrix with only a single row of elements will be spoken of as a *row matrix*.† When it is necessary to write a row matrix at length, the usual square brackets will be employed; but the special brackets $\lfloor\ \rfloor$ will be used to denote a literal row matrix in the abbreviated form. For example,

$$\lfloor y_j \rfloor \equiv [y_1, y_2, y_3].$$

In accordance with the foregoing conventions, the matrix formed from the rth column of an $(\overline{m}, n)$ matrix $[A_{ij}]$ is

$$\{A_{1r}, A_{2r}, \ldots, A_{mr}\},$$

and this can be represented as $\{A_{ir}\}$, provided that i is always taken to be the typical suffix and r the specific suffix. In the same way the matrix formed from the sth row of $[A_{ij}]$ is

$$[A_{s1}, A_{s2}, \ldots, A_{sn}],$$

and this can be expressed as $\lfloor A_{sj} \rfloor$.

* An alternative notation, which is in current use, is $[A]^n_m$.

† A row matrix is often called a *line matrix*, a *vector of the first kind*, or a *prime*; while a column matrix is referred to as a *vector of the second kind*, or a *point*.

The most concise notation for column and row matrices is, as with matrices of a general order, by means of single letters. The particular type of matrix represented by a single letter will always be clear from the context.

(*d*) *Transposition of Matrices.* The *transposed* A' of a matrix A is defined to be the matrix which has rows identical with the columns of A. Thus if $A = [A_{ij}]$, then $A' = [A_{ji}]$. In particular the transposed of a column matrix is a row matrix, and *vice versa.*

In this book an accent applied to a matrix will always denote the transposition of that matrix.

(*e*) *Square, Diagonal, and Unit Matrices.* When the numbers of the rows and columns are equal, say n, the matrix is said to be *square* and of order n: the elements of type A_{ii} then lie in the *principal diagonal.* If all the elements other than those in the principal diagonal are zero, the matrix is called a *diagonal matrix.* The *unit matrix* of order n is defined to be the diagonal matrix of order n which has units for all its principal diagonal elements. It is denoted by I_n, or more simply by I when the order is apparent.*

(*f*) *Symmetrical and Skew Matrices.* When $A_{ij} = A_{ji}$ the matrix A is said to be *symmetrical*, and it is then identical with its transposed. If $A_{ij} = -A_{ji}$, whereas the elements of type A_{ii} are not all zero, the matrix is *skew*; but if both $A_{ij} = -A_{ji}$ and $A_{ii} = 0$ the matrix is *skew symmetric* or *alternate.* Both symmetrical and skew matrices are necessarily square.

(*g*) *Null Matrices.* A matrix of which the elements are all zero is called a *null matrix*, and is represented by 0.

EXAMPLES

(i) $(\overline{3}, 2)$ *Matrix.* $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -1 \end{bmatrix}.$

(ii) *Row Matrix.* $[0, 1, -3, 0].$

(iii) *Column Matrix.* $\{2, -1, -3, 1\}.$

(iv) *Symmetrical Square Matrix.*

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & -2 \end{bmatrix}.$$

* On the Continent the symbol commonly used for the unit matrix is E.

(v) *Diagonal Matrix.* $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

(vi) *Unit Matrix I_2.* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(vii) *Transposed Matrices.*

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -1 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$.

(viii) *Null Matrices.* $\{0, 0, 0\}$; $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

1·3. Summation of Matrices and Scalar Multipliers. Operations with matrices involve operations with the elements of which they are composed. Unless the contrary is stated, these elements will always be understood to be numbers, real or complex, which obey the laws of ordinary algebra (i.e. scalars). It is, however, sometimes useful to consider matrices the elements of which are not ordinary numbers. For example, the elements may themselves be matrices (see § 1·7).

(*a*) *Equality of Matrices.* Equal matrices are necessarily of the same order and have their corresponding elements equal. Thus $A = B$ if $A_{ij} = B_{ij}$.

The equality of two matrices of order m by n implies by definition the satisfaction of mn ordinary equations between their elements. Conversely, a set of ordinary equations can always be represented by a single equation between matrices.

(*b*) *Addition and Subtraction of Matrices.* These operations can only be performed on matrices of the same order.

The *sum* of two such matrices A and B is defined to be the matrix C the typical element C_{ij} of which is $A_{ij} + B_{ij}$. Then

$$A + B = C. \qquad \text{......(1)}$$

The *difference* of A and B is similarly the matrix D the typical element D_{ij} of which is $A_{ij} - B_{ij}$. Then

$$A - B = D. \qquad \text{......(2)}$$

Since any element in a matrix sum is equal to the algebraic sum of the corresponding elements in the summed matrices, we see that the addition of matrices is subject to the same laws as the addition of scalars. Thus the associative and commutative laws of addition hold good.

(c) *Scalar Multipliers.* If $A = B$, it is natural to write equation (1) as $2A = C$, with $C_{ij} = 2A_{ij}$. More generally, the convention is adopted that multiplication of a matrix by a *scalar* coefficient, say l, written either before or after the matrix, is equivalent to multiplication of every element by l. Thus, if

$$lA = Al = C,$$

then

$$C_{ij} = lA_{ij}.$$

The foregoing definitions and conventions are sufficient for the interpretation and reduction of any expression which is linear and homogeneous in a set of matrices of the same order.

Examples

(i) *Matrix Equation Expressed as Scalar Equations.* The single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

yields the four scalar equations

$$a_{11} = b_{11};\; a_{12} = b_{12};\; a_{21} = b_{21};\; a_{22} = b_{22}.$$

(ii) *Scalar Equations Expressed as Matrix Equation.* The four scalar equations

$$a_1 = b_1;\; a_2 = b_2;\; a_3 = b_3;\; a_4 = b_4$$

are contained in each of the matrix equations

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

$$[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4],$$

$$\{a_1, a_2, a_3, a_4\} = \{b_1, b_2, b_3, b_4\}.$$

(iii) *Sum of Matrices.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 7 & 9 & 11 \end{bmatrix}.$$

(iv) *Difference of Matrices.*

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & -7 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 2 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ 1 & -5 & 3 \end{bmatrix}.$$

(v) *Scalar Multipliers.*

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} - 5\begin{bmatrix} -2 & 7 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -31 \\ -4 & 0 \end{bmatrix}.$$

1·4. Multiplication of Matrices. With matrix multiplication two essential facts must be borne in mind. Firstly, matrices are in general not commutative in multiplication. Secondly, two matrices can only be multiplied in a given order provided that a certain condition is satisfied. If the number of columns in B is equal to the number of rows in A, the two matrices are described as *conformable*, and they can then be multiplied in the order $B \times A$. Specifically, if B is a $(\bar{q}, n)$ matrix and A is an $(\bar{n}, p)$ matrix, then the product BA is a $(\bar{q}, p)$ matrix. A scheme which expresses this rule very simply is

$$(\bar{q}, n) \times (\bar{n}, p) = (\bar{q}, p). \qquad \text{......(1)}$$

The product BA is referred to either as A *premultiplied* by B, or as B *postmultiplied* by A.

We may now define the process of multiplication. To obtain the ith element in the jth column of the product $P \equiv BA$, select the ith row of B and the jth column of A, and sum the products of their corresponding elements, beginning at the left-hand end and the top, respectively: thus

$$P_{ij} = \sum_{r=1}^{n} B_{ir} A_{rj}. \qquad \text{......(2)}$$

As a particular case assume in (1) that $q = 1$ and $p = 1$, so that the first matrix has merely a single row, say $\lfloor B_j \rfloor$, of n elements, while the second has a single column, say $\{A_i\}$, of n elements. The product $\lfloor B_j \rfloor \{A_i\}$ in this case is a $(\bar{1}, 1)$ matrix, or a scalar*, which is given by (2) as the sum of the products of the corresponding elements in $\lfloor B_j \rfloor$ and $\{A_i\}$. The general process of multiplication of two matrices may accordingly be interpreted as follows: To obtain the typical element P_{ij} of the product BA, postmultiply the ith row of B by the jth column of A.

From (1) it is seen that two matrices B and A can be multiplied in both the orders BA and AB only provided they are of the types $(\bar{p}, n)$ and $(\bar{n}, p)$: the products are then of the types $(\bar{p}, p)$ and $(\bar{n}, n)$, respectively. In particular this condition is satisfied for square matrices of equal order: however, even in this case the two products are usually not the same. Two matrices having the special property that $BA = AB$ are said to *commute* or to be *permutable*. The unit matrix I, for instance, commutes with any square matrix of the same order.

* The caution should be added that, although a $(\bar{1}, 1)$ matrix can always be treated as a scalar, the converse is only true when conformability allows.

Examples

(i) *Product of Rectangular Matrices.*

$$\begin{bmatrix} 4 & 2 & -1 & 2 \\ 3 & -7 & 1 & -8 \\ 2 & 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ 1 & 5 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (4\times 2)-(2\times 3)-(1\times 1)+(2\times 3), & (4\times 3)+(2\times 0)-(1\times 5)+(2\times 1) \\ (3\times 2)+(7\times 3)+(1\times 1)-(8\times 3), & (3\times 3)-(7\times 0)+(1\times 5)-(8\times 1) \\ (2\times 2)-(4\times 3)-(3\times 1)+(1\times 3), & (2\times 3)+(4\times 0)-(3\times 5)+(1\times 1) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 \\ 4 & 6 \\ -8 & -8 \end{bmatrix}.$$

The rule (1) here gives $(\bar{3}, 4) \times (\bar{4}, 2) = (\bar{3}, 2)$. The matrices are not conformable when taken in the reverse order.

(ii) *Products of Square Matrices.*

$$\begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} (3\times 1)+(4\times 2), & (3\times 2)+(4\times 5) \\ -(2\times 1)-(1\times 2), & -(2\times 2)-(1\times 5) \end{bmatrix} = \begin{bmatrix} 11, & 26 \\ -4, & -9 \end{bmatrix}.$$

When the matrices are multiplied in the reverse order the product is

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} (1\times 3)-(2\times 2), & (1\times 4)-(2\times 1) \\ (2\times 3)-(5\times 2), & (2\times 4)-(5\times 1) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}.$$

Another illustration is provided by the pair of products

$$\begin{bmatrix} 3 & 4 & 2 \\ -2 & -1 & -1 \\ -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 21 & 21 \\ -9 & -9 & -9 \\ -12 & -12 & -12 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 & 2 \\ -2 & -1 & -1 \\ -1 & -3 & -1 \end{bmatrix} = 0.$$

(iii) *Products of Permutable Matrices.*

$$\begin{bmatrix} 0 & -3 & 1 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} 0 & -3 & 1 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(iv) *Column Matrix Premultiplied by Row Matrix.*

$$[5\quad 2\quad -3]\{2\quad -1\quad 4\} \equiv [5\quad 2\quad -3]\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = -4.$$

The rule (1) here gives $(\bar{1}, 3)\times(\bar{3}, 1) = (\bar{1}, 1)$; hence the product is a scalar.

(v) *Row Matrix Premultiplied by Column Matrix.*

$$\{2\quad -1\quad 4\}[5\quad 2\quad -3] \equiv \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}[5\quad 2\quad -3] = \begin{bmatrix} 10 & 4 & -6 \\ -5 & -2 & 3 \\ 20 & 8 & -12 \end{bmatrix}.$$

In this case the rule (1) gives $(\bar{3}, 1)\times(\bar{1}, 3) = (\bar{3}, 3)$. Note that the product here is of a very special type. The elements in any row (or column) are proportional to the corresponding elements in any other row (or column), so that the product has in fact only a single linearly independent row (or column). Examples (ii) and (iii) contain other illustrations of square matrices with this property. More generally, an $(\bar{m}, n)$ matrix with only a single linearly independent row (or column) is always expressible as a product of the form $\{A_i\}\lfloor B_j \rfloor$, where $\{A_i\}$ is a column of m elements and $\lfloor B_j \rfloor$ is a row of n elements. For instance,

$$\begin{bmatrix} 10, & -5, & 20 \\ 4, & -2, & 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}[2, -1, 4] \equiv \{5, 2\}[2, -1, 4].$$

(vi) *Square Matrix Postmultiplied by Column Matrix.*

$$ax \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}.$$

The system of linear algebraic equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2,$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

is accordingly concisely expressible as the matrix equation

$$ax = b.$$

(vii) *Abbreviated Rules for Products of Special Matrices.*

Matrix × column = column,
Row × matrix = row,
Row × column = scalar,
Column × row = matrix with proportional rows and proportional columns.

1·5. Continued Products of Matrices. A continued product of matrices, such as CBA, is to be interpreted as follows. First premultiply A by B, and then premultiply the product BA by C. This process will of course not be possible unless B is conformable with A and C with BA.

In the foregoing definition of a continued product a specific order of multiplication is laid down. However, it will now be shown that the associative law holds good for matrix multiplication, so that the factors in a product may be grouped in any convenient manner, provided that the order of multiplication is not altered.

The associative law requires that if $Y = CB$ and $X = BA$, then

$$CBA = YA = CX. \qquad \text{......(1)}$$

To prove formally that this is a consequence of the definitions of § 1·4, we note firstly that equation (1·4·2) gives for the typical element of X

$$X_{ij} = \sum_{r=1}^{n} B_{ir} A_{rj},$$

where n is the number of columns in B and of rows in A. Hence the (k, j)th element of CX is

$$\sum_{i=1}^{m} C_{ki} X_{ij} = \sum_{i=1}^{m} \sum_{r=1}^{n} C_{ki} B_{ir} A_{rj}, \qquad \text{......(2)}$$

where m is the number of columns in C. Similarly, since

$$Y_{kr} = \sum_{i=1}^{m} C_{ki} B_{ir},$$

the (k, j)th element of YA is

$$\sum_{r=1}^{n} Y_{kr} A_{rj} = \sum_{i=1}^{m} \sum_{r=1}^{n} C_{ki} B_{ir} A_{rj}, \qquad \text{......(3)}$$

in agreement with (2). This proves the truth of (1).

In view of the associative law, it will now be clear that a continued product, or product chain, such as

$$A_m A_{m-1} \dots A_2 A_1$$

will only have a meaning provided that adjacent matrices A_s, A_{s-1} in the chain are conformable. Thus, if r_s and c_s denote, respectively, the number of rows and columns in A_s, the conditions to be satisfied are

$$c_s = r_{s-1}$$

for $s = m, m-1, \dots, 2$. The scheme of multiplication in this case may be represented by

$$(\bar{r}_m, r_{m-1}) \times (\bar{r}_{m-1}, r_{m-2}) \times \dots \times (\bar{r}_2, r_1) \times (\bar{r}_1, c_1) = (\bar{r}_m, c_1).$$

The product, accordingly, is a matrix having r_m rows and c_1 columns.

For brevity, the product of two equal square matrices A is written A^2, and a similar notation is adopted for the other positive integral powers.

The distributive law also holds good for matrix multiplication. For example,

$$E(A+B)F = EAF + EBF. \qquad \text{......(4)}$$

The correctness of this follows at once from the formula (2) and the definition of addition.

Examples

(i) *Associative Law.* Compare

$$\begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 26 \\ -4 & -9 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 22 & 67 \\ -8 & -23 \end{bmatrix} \qquad \text{......(5)}$$

with

$$\begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 4 & 13 \end{bmatrix} = \begin{bmatrix} 22 & 67 \\ -8 & -23 \end{bmatrix}. \qquad \text{......(6)}$$

(ii) *Rule for Product.* With

$$P = \{2 \quad -1 \quad 4\}[5 \quad 2 \quad -3]\{1 \quad 0 \quad 2\}[-1 \quad 2], \qquad \text{......(7)}$$

the rule for the product gives

$$(\bar{3}, 1) \times (\bar{1}, 3) \times (\bar{3}, 1) \times (\bar{1}, 2) = (\bar{3}, 2).$$

The product thus exists, and is a matrix with three rows and two columns. To evaluate P, note that the part-product $[5 \quad 2 \quad -3]\{1 \quad 0 \quad 2\}$ yields the scalar -1, which may be brought to the front: hence

$$P = -1 \times \{2 \quad -1 \quad 4\}[-1 \quad 2] = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 4 & -8 \end{bmatrix}.$$

(iii) *Positive Powers of a Square Matrix.*

$$\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}^2 \equiv \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 16 \\ -8 & -7 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}^3 = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 16 \\ -8 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 16 \\ -8 & -7 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -29 & 20 \\ -10 & -39 \end{bmatrix}.$$

Note that positive integral powers of a square matrix are permutable.

(iv) *Computation from "Right to Left".* In this method the complete product is evaluated by successive premultiplications. For instance,