# Introduction

As the title *Entropy, compactness and the approximation of operators* suggests, this book is about entropy, compactness and approximation properties of linear and continuous operators acting between Banach spaces. This indicates that the reader is first of all supposed to be aquainted with the notion of *a Banach space* and the notion of a *linear and continuous operator* $T:E \to F$ from a Banach space $E$ into a Banach space $F$. These two notions are closely related to each other.

A *norm* $\|\cdot\|_0$ on a Banach space $E$ is said to be *equivalent* to the original norm $\|\cdot\|$ on $E$, if there exist constants $c > 0$ and $C > 0$ such that

$$c \cdot \|x\| \leqslant \|x\|_0 \leqslant C \cdot \|x\| \quad \text{for all } x \in E.$$

However, instead of assigning another norm $\|x\|_0$ to the same element $x \in E$ we can also regard $x$ with the new norm $\|x\|_0$ as an element $y = Sx$ of another Banach space $E_0$. The map $S:E \to E_0$ defined in this way is a linear and continuous operator from $E$ onto $E_0$ with a continuous inverse $S^{-1}:E_0 \to E$. An operator with these properties is called an *isomorphism.* The corresponding Banach spaces $E$ and $E_0$ are said to be *isomorphic.*

Among the examples of Banach spaces to appear in this book the *Banach spaces* $C(X)$ of continuous functions on a compact metric space $X$ and *Hilbert spaces* will take a primary place. In addition the reader is presented with the *spaces* $l_p$ of $p$-summable sequences, with the corresponding *spaces* $L_p(X, \mu)$ of functions $f$ on a compact metric space $X$ whose $p$th power $|f|^p$ is integrable with respect to a Borel measure $\mu$ on $X$, $1 \leqslant p < \infty$, as well as with the *spaces* $l_\infty$ *and* $c_0$ of bounded sequences and null sequences, respectively, and with the *space* $L_\infty(X, \mu)$ of $\mu$ essentially bounded functions on $X$.

Special linear and continuous operators that will be used are *diagonal operators* $D:l_p \to l_p$ acting in a sequence space $l_p$, $1 \leqslant p \leqslant \infty$, and *integral operators* from $L_p(X, \mu)$ into $C(X)$ as well as from $C(X)$ into itself.

Given a linear operator $T$ from a Banach space $E$ into a Banach space $F$ the question of continuity of $T$ is normally decided by checking the boundedness of $T$. Indeed, a linear operator $T:E \to F$ is continuous if and only if it is bounded. If $T$ is known from the very beginning to be linear and either continuous or bounded we shall in general omit these two adjectives and simply use the notation *operator*. The class of all operators

$T:E\to F$ is denoted by $L(E,F)$. When equipped with the *operator norm*

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\|$$

the class $L(E,F)$ becomes a Banach space itself.

Compactness properties of operators give rise to subclasses of the class $L(E,F)$. The main intention of the book is to quantify the '*degree*' *of compactness* of an operator $T:E\to F$ and to study its relation to other analytical properties of $T$. Among these, *approximability of T by finite rank operators*, a subject dealt with in chapter 2, plays a decisive role.

An operator $T:E\to F$ is called a *finite rank operator*, if its range $R(T)$ is a finite-dimensional subspace of $F$. If either the Banach space $E$ or the Banach space $F$ is finite-dimensional, the operator $T$ is called *finite-dimensional.*

We recall that any finite-dimensional subspace of a Banach space is closed. In contrast with that an infinite-dimensional linear subspace of a Banach space need not be closed. The subspaces of Banach spaces occurring in the context of this book will in general be both linear and closed. Therefore we shall omit these two adjectives if no confusion is possible.

The *range* $R(T)$ of an operator $T:E\to F$ is a linear subspace of $F$ which is not necessarily closed. In order to have the chance of using a closed linear subspace related to the range $R(T)$ of $T$ we take the closed hull $\overline{R(T)}$ of $R(T)$ if this turns out to be advantageous, for instance for obtaining the *canonical factorization*

$$T = I_{F_0}^F T_0$$

of $T$ through the Banach space $F_0 = \overline{R(T)}$. The operator $T_0:E\to F_0$ then is defined by

$$T_0x = Tx \quad \text{for } x\in E$$

and is called *the operator induced by T*, while $I_{F_0}^F:F_0\to F$ is understood to be the *natural* or *canonical embedding* of the 'subspace' $F_0$ of $F$ into $F$.

Another *canonical factorization* of an operator $T:E\to F$ refers to the null space $N=N(T)$ of $T$. By the continuity of $T$, the *null space*

$$N(T) = \{x\in E: Tx=0\}$$

or the *kernel* of $T$, as it is sometimes called, must be a closed subspace of $E$. This implies that the *quotient space* $E/N$ is complete with respect to the norm

$$\|\bar{x}\| = \inf\{\|x-z\|: z\in N\}$$

of its elements

$$\bar{x} = \{x-z: z\in N\},$$

which are called the *cosets* of the elements $x \in E$ with respect to the subspace $N \subseteq E$. The operator $Q_N^E: E \to E/N$ defined by

$$Q_N^E x = \bar{x}$$

is referred to as the *natural* or the '*canonical*' *surjection* of $E$ onto the quotient space $E/N$. Quite often we shall also use the expression *quotient map*.

The canonical factorization connected with the null space $N = N(T)$ of $T: E \to F$ that we have in mind is the factorization

$$T = T_0 Q_N^E$$

of $T$ over the quotient space $E/N$. The operator $T_0: E/N \to F$ in this situation is defined by

$$T_0 \bar{x} = Tx \quad \text{for } \bar{x} \in E/N$$

and also called *the operator induced by* $T$. Note that the definition of $T_0$ in fact makes sense since the value $T_0 \bar{x}$ in $F$ is independent of the choice of a representative $x$ in the coset $\bar{x}$.

Besides the two kinds of canonical factorizations, which apply to arbitrary operators $T: E \to F$, we have specific representations for special operators $T: E \to F$. In particular, if $T$ is a finite rank operator, $T$ can be represented as a finite sum of rank 1 operators. Since a rank 1 operator $T: E \to F$ has a one-dimensional range it allows a representation

$$Tx = A(x) \cdot y_0 \quad \text{for } x \in E$$

with an element $y_0 \in F$ and an operator $A$ from $E$ into the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. Operators of this kind will be called *functionals* and from now on will be denoted by lower case latin letters $a, b, c, \ldots$ This notation expresses a kind of similarity between elements $x \in E$ and functionals $a$ over $E$. The notation $\langle x, a \rangle$ for the value of the functional $a$ on the element $x \in E$ emphasizes this idea. It encourages the reader to fix $x$ in $E$ and to let $a$ vary in the Banach space $\boldsymbol{L}(E, \mathbb{R})$ of all linear and continuous functionals over $E$. Indeed, according to what has been said about $\boldsymbol{L}(E, F)$, the linear space $\boldsymbol{L}(E, \mathbb{R})$ is a Banach space with respect to the norm

$$\|a\| = \sup_{\|x\| \leqslant 1} |\langle x, a \rangle|.$$

It is denoted by $E'$ and called *the dual of* $E$. As indicated by the notation $\langle x, a \rangle$ any $x \in E$ can be considered as a functional over $E'$. This functional is obviously linear and, moreover, even continuous, its norm being given by

$$\|x\| = \sup_{\|a\| \leqslant 1} |\langle x, a \rangle|,$$

that is by the norm of $x$ as an element of $E$. This is a consequence of the famous *Hahn–Banach extension theorem*. The operator that assigns to each $x \in E$ the corresponding functional over $E'$ will be denoted by $K_E: E \to E''$ and called *the canonical embedding of* $E$ *into its bidual* $E'' = (E')'$.

The agreement on the notation of functionals over a Banach space $E$ secures the possibility of a representation of an operator $A: E \to F$ with rank $(A) \leqslant n$ by a finite sum

$$Ax = \sum_{i=1}^{n} \langle x, a_i \rangle y_i$$

with elements $y_i \in F$ and functionals $a_i \in E'$.

Quite often the investigation of finite rank operators is the basis for the investigation of infinite-dimensional operators, in so far as an infinite-dimensional operator $T: E \to F$ is approximated by finite rank operators $A: E \to F$. In the case of an operator $T: E \to C(X)$ with values in a Banach space $C(X)$ of continuous functions the degree of approximability of $T$ by finite rank operators is essentially determined by the degree of compactness of the underlying compact metric space $X$. This is proved in detail in chapter 5, section 5.6. On the other hand, one of the central results of the book says that a certain degree of approximability of any operator $T: E \to F$ by finite rank operators implies a certain degree of compactness of $T$. The precise derivation of this claim is given in section 3.1, 'Inequalities of Bernstein type'.

Another central result concerns the influence of the degree of compactness of an operator $T: E \to E$ on the *rate of decrease of the eigenvalue sequence* $\lambda_1(T), \lambda_2(T), \ldots, \lambda_n(T), \ldots$ The inequalities expressing the corresponding relation between compactness properties and spectral properties of operators $T: E \to E$ are proved in the framework of 'A refined Riesz theory' (chapter 4, section 4.2). They enable us to predict certain summability properties of the eigenvalues $\lambda_n(T)$ under appropriate suppositions about the degree of compactness of $T$.

But how should we quantify the degree of compactness of operators $T: E \to F$? Among various possibilities for doing this there is a predestinate one which refers to *entropy quantities* of $T$. What are they?

In chapter 1, sections 1.1 and 1.2, entropy quantities are first introduced for bounded subsets of metric spaces. In the subsequent sections 1.3 and 1.4 these concepts are transferred to operators $T \in L(E, F)$.

The *notion of a metric space*, which is basic for the beginning of the book, is also a central notion for chapter 5 devoted to *operators* $T: E \to C(X)$ *with values in a Banach space* $C(X)$ of continuous functions on a compact metric space $X$. As already mentioned, the degree of compactness of $X$ or, in other words, the entropy properties of $X$, imply a certain degree of approximability of $T$ by finite rank operators and hence, because of the Bernstein type inequalities, also certain entropy properties of $T$. This fact alone would justify the investigation of operators $T: E \to C(X)$ in a separate chapter. But in addition it turns out that *compactness properties of* $C(X)$*-valued operators* are in a sense *representatives for compactness properties of operators* between arbitrary Banach spaces. This will be pointed out in section 5.1.

The book ends with chapter 6, 'Operator theoretical methods in the local theory of Banach spaces'. This final chapter seeks to demonstrate that the theory of Banach spaces, which seems to take priority over the theory of operators in Banach spaces, at least up to a certain extent, can also be developed on the basis of the operator theory.

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# 1
# *Entropy quantities*

Entropy quantities, in particular entropy numbers, in their proper sense, are set functions defined for bounded subsets of metric spaces. The theory of linear and continuous operators between Banach spaces involves the consideration of bounded subsets in Banach spaces. Indeed, the image $T(U_E)$ of the closed unit ball $U_E$ of a Banach space $E$ under a linear and continuous operator $T$ from $E$ into a Banach space $F$ is a bounded subset of $F$ and thus possesses well-defined entropy numbers. Having observed this, one can conjecture that entropy numbers might be useful tools for investigating properties of linear and continuous operators between Banach spaces. They *are* in fact and, as we shall see later, in particular prove to be appropriate quantities for reflecting compactness and spectral properties of operators.

## 1.1. Entropy numbers of sets

Let $(X,d)$ be a metric space and $x_0 \in X$. Then we denote by

$$\mathring{U}(x_0;\varepsilon)=\{x\in X: d(x,x_0)<\varepsilon\}$$

the open ball of radius $\varepsilon>0$ with centre $x_0$, and by

$$U(x_0;\varepsilon)=\{x\in X: d(x,x_0)\leqslant\varepsilon\}$$

the corresponding closed ball. A subset $M\subseteq X$ is said to be *bounded* if it is contained in an appropriate ball $U(x_0;\varepsilon)$.

Bounded subsets of metric spaces give rise to covering problems as well as packing problems. The *coverings* we are thinking of are coverings by closed balls $U(x_i;\varepsilon)$ of uniform radius $\varepsilon$ so that

$$M\subseteq\bigcup_{i\in I}U(x_i;\varepsilon). \tag{1.1.1}$$

A system of points $\{x_i\}_{i\in I}$ producing a covering (1.1.1) of $M$ in $X$ is called an *$\varepsilon$-net for $M$ in $X$*. By a *packing* of a bounded subset $M\subseteq X$ we mean a system of points $x_i\in M$, $i\in I$, such that

$$d(x_i,x_k)>2\rho \quad \text{for } i\neq k \text{ and all } i,k\in I, \tag{1.1.2}$$

where $\rho$ is a certain positive number. A system $\{x_i\}_{i\in I}$ with the property (1.1.2) is also called a *$\rho$-distant subset of $M\subseteq X$*.

 

There is a close relation between covering and packing properties of bounded subsets in metric spaces (cf. Kolmogorov and Tichomirov 1959; Mitjagin 1961; Mitjagin and Pelczyński 1966; Lorentz 1966; Triebel 1970; Pietsch 1978). In order to reveal this relation, we introduce entropy numbers $\varepsilon_n(M)$ and capacity numbers or so-called inner entropy numbers $\varphi_n(M)$ for bounded sets $M \subseteq X$. The *nth entropy number* $\varepsilon_n(M)$ is defined by

$$\varepsilon_n(M) = \inf\{\varepsilon > 0\text{: there exists an } \varepsilon\text{-net for } M \text{ in } X \text{ consisting of } q \leqslant n \text{ points}\}.$$

Of course, one can also operate with coverings of $M$ by open balls instead of closed ones, and characterize $\varepsilon_n(M)$ as

$$\varepsilon_n(M) = \inf\left\{\varepsilon > 0\text{: there exist } q \leqslant n \text{ points } x_1, x_2, \ldots, x_q \text{ in } X \text{ such that } M \subseteq \bigcup_{i=1}^{q} \mathring{U}(x_i; \varepsilon)\right\}.$$

The *nth inner entropy number* $\varphi_n(M)$ is defined by

$$\varphi_n(M) = \sup\{\rho > 0\text{: there exists a } \rho\text{-distant subset of } M \text{ consisting of } p > n \text{ points}\}$$

or, equivalently,

$$\varphi_n(M) = \sup\{\rho > 0\text{: there exist } p > n \text{ points } x_1, x_2, \ldots, x_p \text{ in } M \text{ such that } d(x_i, x_k) \geqslant 2\rho \text{ for } i \neq k\}.$$

If $M$ consists of less than $n + 1$ elements we put

$$\varphi_n(M) = 0.$$

Given $\varepsilon > \varepsilon_n(M)$ and $\rho$ with $0 < \rho < \varphi_n(M)$ we can find points $y_1, y_2, \ldots, y_q$ in $X$ and points $x_1, x_2, \ldots, x_p$ in $M$ with $q \leqslant n$ and $p > n$ such that

$$M \subseteq \bigcup_{j=1}^{q} U(y_j; \varepsilon) \text{ and } d(x_i, x_k) > 2\rho \quad \text{for } i \neq k,\ 1 \leqslant i, k \leqslant p.$$

Since $p > n \geqslant q$ at least one ball $U(y_j; \varepsilon)$ must contain two of the elements $x_i \in M$, say

$$x_i \in U(y_j; \varepsilon) \quad \text{and } x_k \in U(y_j; \varepsilon).$$

Accordingly, we have

$$d(x_i, x_k) \leqslant d(x_i, y_j) + d(y_j, x_k) \leqslant 2\varepsilon.$$

Hence it follows that $2\rho < 2\varepsilon$ which implies

$$\varphi_n(M) \leqslant \varepsilon_n(M). \tag{1.1.3}$$

On the other hand, for any $\rho$ with $\varphi_n(M) < \rho$ there exists a maximal $\rho$-distant subset $\{x_1, x_2, \ldots, x_q\}$ of $M$ with $q \leqslant n$. The maximality of the subset $\{x_1, x_2, \dots, x_q\}$ amounts to the fact that one can assign to any $x \in M$ at least one element $x_i$ such that

$$d(x, x_i) \leqslant 2\rho.$$

Thus

$$M \subseteq \bigcup_{i=1}^{q} U(x_i; 2\rho)$$

turns out to be true which shows that $\varepsilon_n(M) \leqslant 2\rho$. The final result is

$$\varepsilon_n(M) \leqslant 2\varphi_n(M). \tag{1.1.4}$$

A subset $M$ of a metric space $X$ is called *precompact* if for every $\varepsilon > 0$ there exists a finite $\varepsilon$-net for $M$. The minimal number $m = N(M; \varepsilon)$ of elements $x_1, x_2, \dots, x_m$ in $X$ forming an $\varepsilon$-net for $M$ is called *the entropy function of the precompact set* $M$. If we are given an arbitrary finite $\varepsilon$-net $\{x_1, x_2, \dots, x_n\}$ of $M$ so that

$$M \subseteq \bigcup_{i=1}^{n} U(x_i; \varepsilon),$$

we may conclude that $n \geqslant N(M; \varepsilon)$ and

$$\varepsilon_n(M) \leqslant \varepsilon \quad \text{for all } n \geqslant N(M; \varepsilon).$$

Hence we have

$$\lim_{n \to \infty} \varepsilon_n(M) = 0 \tag{1.1.5}$$

for every precompact set $M \subseteq X$. Conversely, the condition (1.1.5) proves to be sufficient for the precompactness of the set $M$. An equivalent characterization of precompact sets $M \subseteq X$ says that

$$\lim_{n \to \infty} \varphi_n(M) = 0. \tag{1.1.6}$$

Since the sequences $\varepsilon_n(M)$ and $\varphi_n(M)$ are monotonously decreasing, the rate of decrease may be regarded as a measure for the degree of precompactness of the set $M$.

In a finite-dimensional Banach space every bounded subset is precompact. This can easily be seen by estimating the entropy numbers of the unit ball of a Banach space $E$ with $\dim(E) = m$ from above. Let

$$U_E = \{x \in E : \|x\| \leqslant 1\}$$

stand for the closed unit ball of $E$ and

$$\mathring{U}_E = \{x \in E : \|x\| < 1\}$$

for the open one. We refer to the inner entropy numbers $\varphi_n(U_E)$ of $U_E$, fix an arbitrary positive number $\rho < \varphi_n(U_E)$, and determine a system of $p > n$ elements $x_1, x_2, \dots, x_p$ in $U_E$ with

$$\|x_i - x_k\| > 2\rho \quad \text{for } i \neq k. \tag{1.1.7}$$

Then we consider the closed balls

$$U(x_i;\rho)=\{x_i+\rho U_E\}.$$

Since

$$\rho<\varphi_n(U_E)\leqslant\varphi_1(U_E)=1,$$

we have

$$\|x\|\leqslant\|x-x_i\|+\|x_i\|\leqslant\rho+1<2\quad\text{for } x\in U(x_i;\rho)$$

and, furthermore,

$$U(x_i;\rho)\cap U(x_k;\rho)=\varnothing\quad\text{for } i\neq k$$

by (1.1.7). Let us regard $U(x_i;\rho)$, $1\leqslant i\leqslant p$, as a system of $p$ non-intersecting closed subsets of the $m$-dimensional euclidean space $l_2^m$. Since these subsets are contained in the closed subset $2U_E$ we may use the Lebesgue measure on $l_2^m$ and carry out a comparison of volumes, namely

$$\sum_{i=1}^{p}\text{vol}_m(U(x_i;\rho))\leqslant\text{vol}_m(2U_E)$$

which amounts to

$$p\cdot\rho^m\cdot\text{vol}_m(U_E)\leqslant 2^m\,\text{vol}_m(U_E).$$

Because $\text{vol}_m(U_E)>0$ and $p>n$ we may conclude that

$$\rho\leqslant 2\cdot n^{-1/m}.$$

This yields

$$\varphi_n(U_E)\leqslant 2\cdot n^{-1/m} \tag{1.1.8}$$

and thus confirms the statement $\lim_{n\to\infty}\varphi_n(U_E)=0$. But, what is more, the sequence $n^{-1/m}$ exactly reflects the asymptotic behaviour of the sequences $(\varphi_n(U_E))$ and $(\varepsilon_n(U_E))$. Indeed, given $\varepsilon>\varepsilon_n(U_E)$ there exist $q\leqslant n$ elements $y_1,y_2,\ldots,y_q$ in $E$ such that

$$U_E\subseteq\bigcup_{i=1}^{q}\{y_i+\varepsilon U_E\}.$$

This time a comparison of volumes in the $m$-dimensional euclidean space leads us to

$$\text{vol}_m(U_E)\leqslant q\cdot\varepsilon^m\,\text{vol}_m(U_E)$$

and finally yields

$$\varepsilon_n(U_E)\geqslant n^{-1/m}. \tag{1.1.9}$$

Combining (1.1.9), (1.1.4), and (1.1.8) we recognize that

$$n^{-1/m}\leqslant\varepsilon_n(U_E)\leqslant 4\cdot n^{-1/m}\quad\text{for }\dim(E)=m. \tag{1.1.10}$$

Let us emphasize that the asymptotic behaviour of the entropy numbers $\varepsilon_n(U_E)$ is essentially determined by the dimension $m$ of the underlying Banach space $E$.

So far $E$ has been tacitly supposed to be a real $m$-dimensional Banach space. In the case of a complex Banach space of dimension $m$ the comparison of volumes takes place in a real euclidean space of dimension $2m$. Correspondingly, (1.1.10) has to be replaced by

$$n^{-1/2m} \leqslant \varepsilon_n(U_E) \leqslant 4 \cdot n^{-1/2m} \tag{1.1.11}$$

## 1.2. Entropy moduli of sets

The comparison of volumes just carried out for coverings of the unit ball $U_E$ of a finite-dimensional Banach space $E$ opens new perspectives for estimating the degree of precompactness even in the general situation of a bounded subset in an arbitrary Banach space.

For the moment let $M \subset E$ be a bounded and Lebesgue-measurable subset of a Banach space $E$ with $\dim(E) = m$. This means that the set $M$ is measurable in the Lebesgue sense when it is considered as a subset of the $m$-dimensional euclidean space $l_2^m$. The unit ball $U_E$ of the $m$-dimensional Banach space $E$ always has this property. A covering

$$M \subseteq \bigcup_{i=1}^{k} \{x_i + \varepsilon U_E\}$$

of the set $M$, in a similar way as in the case $M = U_E$, then gives rise to an inequality

$$\mathrm{vol}_m(M) \leqslant k\varepsilon^m \, \mathrm{vol}_m(U_E) \tag{1.2.1}$$

between the volume $\mathrm{vol}_m(M)$ of the set $M$ and the volume $\mathrm{vol}_m(U_E)$ of the unit ball $U_E$. Replacing $\varepsilon$ by the corresponding infimum $\varepsilon_k(M)$ we obtain the inequality

$$\left(\frac{\mathrm{vol}_m(M)}{\mathrm{vol}_m(U_E)}\right)^{1/m} \leqslant k^{1/m}\varepsilon_k(M). \tag{1.2.2}$$

To obtain an optimal estimation of the so-called *volume ratio* $(\mathrm{vol}_m(M)/\mathrm{vol}_m(U_E))^{1/m}$ from above by the entropy quantities $k^{1/m}\varepsilon_k(M)$ we take the infimum with respect to $k$ on the right-hand side of (1.2.2). Then

$$\left(\frac{\mathrm{vol}_m(M)}{\mathrm{vol}_m(U_E)}\right)^{1/m} \leqslant \inf_{1 \leqslant k < \infty} k^{1/m}\varepsilon_k(M) \tag{1.2.3}$$

appears. Now we observe that the expression

$$g_n(M) = \inf_{1 \leqslant k < \infty} k^{1/n}\varepsilon_k(M), \quad n = 1, 2, 3, \ldots, \tag{1.2.4}$$

makes sense for any bounded subset $M$ of an arbitrary Banach space $E$. It is called *the nth entropy modulus of the set M* (cf. Carl 1982, 1984). The asymptotic behaviour of the sequence of entropy moduli $(g_n(M))$ again is a criterion for the degree of precompactness of the underlying set $M$.