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95 Nonlinear superposition operators

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Preface

The present monograph is concerned with a thorough study of the nonlinear operator

$$Fx(s) = f(s, x(s)). \tag{1}$$

Here $f = f(s, u)$ is a given function which is defined on the Cartesian product of some set Ω, which in most cases is either a metric space or a measure space or both, with the set $\mathbb{R}$ of real or the set $\mathbb{C}$ of complex numbers, and takes values in $\mathbb{R}$ or $\mathbb{C}$, respectively. By definition, the operator F associates to each real (or complex) function $x(s)$ on Ω the real (or complex) function $f(s, x(s))$ on Ω; therefore F is usually called a *superposition operator* (sometimes also *composition operator*, *substitution operator*, or *Nemytskij operator*).

In an implicit form, the superposition operator (1) can be found in the first pages of any calculus textbook (in the old terminology, as "composite function", "function of a function", etc.), where some of its elementary properties are described. Typical examples of such properties are the continuity of the superposition of continuous functions, the differentiability of the superposition of differentiable functions, and similar statements. Many other results of this type are scattered, mostly as lemmas or auxiliary results, in a vast literature on *mathematical analysis*, *functional analysis*, *differential* and *integral equations*, *probability theory* and *statistics*, *variational calculus*, *optimization theory*, and other fields of contemporary mathematics – the superposition operator occurs everywhere.

In many situations, the investigation of the basic properties of the operator (1) is quite straightforward and does not involve any particular difficulties. But this is not always so. In fact, at the beginning of nonlinear analysis it was often tacitly assumed that "nice" properties of the function f carry over to the corresponding operator F; this turned out to be false even in well-known classical function spaces. A typical example of this phenomenon is the behaviour of the superposition operator in Lebesgue spaces. For instance, the smoothness (and even the analyticity) of the function f does not imply the smoothness of F, considered as an operator between two Lebesgue spaces. Moreover, just the fact that F acts from the Lebesgue space L_p into the Lebesgue space L_q, say, leads to the very restrictive growth condition $f(s, u) = O(|u|^{p/q})$. Further, if F is (Fréchet-) differentiable between L_p and L_q, and the partial derivative f'_u of f with respect to u exists, then necessarily $f'_u(s, u) = O(|u|^{(p-q)/q})$ if $p \geq q$, and $f'_u(s, u) \equiv 0$ if $p < q$. Finally, if F is analytic between L_p and L_q, then the function f reduces to a polynomial in u (of degree at most p/q). All these facts are rather surprising; they show that many of the important properties of the function f do not imply analogous properties of the operator F, or vice versa.

Classical mathematical analysis mainly dealt with *spaces of continuous* or *differentiable functions*; already *Lebesgue spaces* arose only in special fields, e.g. Fourier series, approximation theory, probability theory. In modern nonlinear analysis, however, the arsenal of available function spaces has been considerably enlarged. In this connection, one should mention *Sobolev spaces* and their generalizations which are simply indispensable for the study of partial differential equations, *Orlicz spaces* which are the natural tool in the theory of both linear and nonlinear integral equations, *Hölder spaces* and their generalizations which are basic for the investigation of singular integral equations, *Lorentz* and *Marcinkiewicz spaces* which are widely used in interpolation theory for linear operators, and special classes of *spaces of differentiable* or *smooth functions* which frequently occur in the theory of ordinary or partial differential equations and variational calculus. The usefulness of all these spaces in various fields of mathematical analysis emphasizes the need for a systematic study of the superposition operator (1), considered as an operator from one such space into another.

In this connection, there are still many open problems. In particular, for many of these spaces one does not even know *acting conditions* for F, by means of conditions on f, which are both necessary and sufficient (sufficient conditions are often easily formulated). On the other hand, many special facts regarding the elementary properties of F, such as *continuity*, *boundedness*, or *compactness*, are well-known in, say, Orlicz spaces, Hölder spaces, or Sobolev spaces. Unfortunately, all these results are scattered in research papers and special monographs. We therefore conclude that it would be useful to collect the basic facts on the superposition operator, to present the main ideas which have been shown to be useful in studying its properties, and to provide a comparison of its behaviour in different spaces. This is the purpose of the present monograph.

Here the key problem is, as already mentioned, to find conditions on the function f which imply certain properties of the corresponding operator F. In this connection, the main properties we are interested in are: *boundedness* and *compactness* on certain subsets, *continuity* and *differentiability* at single points, *continuity* and *continuous differentiability* on open subsets, *special continuity properties* (such as *Lipschitz*, *uniform*, or *weak continuity*), *analyticity*, and related properties. These are just the properties which occur most frequently in the application of methods of nonlinear analysis, such as *fixed-point principles*, *degree theory*, *bifurcation methods*, *variational techniques*, to nonlinear equations involving superposition operators. Thus, the reader may typically find answers to questions of the following type: what are necessary and sufficient conditions for the function f such that the corresponding operator F maps the Lebesgue space L_p into the Lebesgue space L_q, or is continuous between two Orlicz spaces L_M and L_N, or differentiable between two Hölder spaces H_ϕ and H_ψ, or bounded between two Sobolev spaces W_p^k and W_q^m, or Lipschitz continuous in the space BV?

When preparing the material for this monograph, we intentionally confined ourselves to the *scalar case*. The vector case, i.e. when the superposition operator F maps $\mathbb{R}^m$- (or $\mathbb{C}^m$-) valued functions on Ω into $\mathbb{R}^n$- (or $\mathbb{C}^n$-) valued functions on Ω (and f is defined, of course , on $\Omega \times \mathbb{R}^m$ or $\Omega \times \mathbb{C}^m$ with values in $\mathbb{R}^n$ or $\mathbb{C}^n$, respectively), is at least as important as the scalar case. However, much less is known in this case, and the development of a "higher-dimensional" theory would be beyond the scope of the present work and would probably require us to increase the size of this survey at least twofold. In large parts of the monograph, Ω may also be the set of all natural numbers, equipped with the counting measure; consequently, our results cover superposition operators in sequence spaces as well. The main emphasis is put, however, on "usual" functions, i.e. the case when Ω is some domain in Euclidean space.

Apart from the superposition operator (1), the related operator

$$\Phi x(s) = x(\phi(s)) \tag{2}$$

is sometimes also called superposition operator in the literature, where ϕ is some bijection of Ω onto itself; more precisely, operators of this type should be called *"inner" superposition operators*, in contrast to the "outer" superposition operator (1). In spite of the similar structure of the operators (1) and (2), their properties are quite different; this is clear, for instance, from the fact that the operator (2) is *linear*, while the major difficulty in the study of the operator (1) lies in its *nonlinearity*. Throughout this monograph, we shall be concerned only with the outer superposition operator (1).

Another operator which is closely related to the operator (1) is the *integral functional*

$$\Phi x = \int_\Omega f(s, x(s))ds\,, \tag{3}$$

which is of fundamental importance, for example, in *variational problems* of nonlinear analysis. We shall be concerned with the operator (3) only marginally and refer to the vast literature on variational methods.

The monograph consists of nine chapters. Each chapter is divided into a number of sections and provides a self-contained systematic study of the superposition operator in some class of function (or sequence) spaces. We have tried to make the exposition as complete and explicit as possible, including proofs, examples, and counterexamples. The last section of each chapter is devoted to possible generalizations, special cases, open problems, related fields and detailed bibliographical references. Each theorem, lemma, or formula is indexed within the corresponding chapter; thus, for example, Lemma 1.2 is the second lemma of the first chapter. By $\Rightarrow$ and $\Leftarrow$ we denote the beginning and the end, respectively, of a proof.

The contents of the monograph go as follows. The first chapter is entirely devoted to the study of the superposition operator in the *space* $S = S(\Omega)$ *of measurable functions* on Ω, where Ω is an arbitrary nonempty set with measure. Here a basic problem is that of finding conditions on the function f which ensure that the operator F maps measurable functions into measurable functions. Surprisingly enough, this turns out to be a highly nontrivial problem.

As a matter of fact, the space S is a complete metric linear space, but not normable. Most fundamental principles of linear and nonlinear functional analysis are formulated, however, in a Banach space setting. Consequently, it is desirable to study the properties of the superposition operator not only in S, but also in normed subspaces of S. It turns out that the most appropriate class of Banach spaces of measurable functions is that of so-called *ideal spaces* (or *Banach lattices*), which were considered by many authors for different purposes. General properties of the superposition operator in ideal spaces are described in detail in the second chapter.

The third and fourth chapters are concerned with the superposition operator in *Lebesgue* and *Orlicz spaces*, respectively. Here the theory is most complete and advanced, and one can characterize all basic properties of the operator F (in particular, acting conditions) in terms of the generating function f.

Some other classes of ideal spaces which include, for example, the classical *Lorentz* and *Marcinkiewicz spaces* are dealt with in the fifth chapter. In this connection, only very few elementary results are presently known.

The sixth chapter is devoted to the superposition operator in the *space* $C = C(\Omega)$ *of continuous functions* on Ω, where Ω is a compact domain without isolated points in Euclidean space. Here the basic facts are well-known "folklore"; however, we shall also discuss some special problems which have not been studied yet. Moreover, we briefly discuss the superposition operator in the *space* BV *of functions of bounded variation.*

In the seventh chapter we shall present a systematic study of the superposition operator in *Hölder-type spaces.* It turns out that the behaviour of the superposition operator in such spaces is quite different from that in spaces of measurable functions.

The eighth chapter will be concerned with the superposition operator in spaces of functions which are characterized by certain differentiability or smoothness properties. Moreover, we shall consider the operator F in various *spaces of finitely* or *infinitely differentiable functions*, including *Roumieu*, *Beurling* and *Gevrey classes.*

Some results on the superposition operator in *Sobolev spaces* are given in the ninth chapter. Unfortunately, in spite of the importance of these spaces in the theory of distributions and partial differential equations, they have been given very little attention in the literature.

Some remarks on the bibliography are in order. We hope to present a rather exhaustive list of references on the superposition operator in function and sequence spaces. The bibliography at the end covers the period from 1918 to 1988 and contains about 400 items, half of them in Russian; thus, it may also serve as a guide to the Soviet literature. For the reader's convenience, we have added English translations (if there are any) of Russian books and major journal papers, and, beginning with 1960, the corresponding review numbers of Zentralblatt für Mathematik (Zbl.), Referativnyj Zhurnal Matematika (R.Zh.), and Mathematical Reviews (M.R.). We are indebted to Nguyêñ Hôǹg Thái, Heinz-Willi Kröger and Reiner Welk for computer-aided help in finding many review numbers.

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