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62. Injective modules

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To our parents

Preface

In this book, we make no claim to give an exhaustive treatment of the theory of injective modules. Our aim has been to see how injective modules may be used in the context of commutative algebra and how, by means of injective modules, the results of commutative algebra may be generalized to apply in a non-commutative setting. The possibility of the use of injective modules in this way was opened up by the fundamental work of E. Matlis in his paper 'Injective modules over Noetherian rings'. Some of the results in this book can be obtained by more elementary means, without the use of injective modules. But, in our view, a good working knowledge of injective modules is a sound investment for module theorists.

It is extremely tempting nowadays to do everything in a general Abelian category instead of in the category of modules, and indeed most of what has been done here fits well into that more general setting. We have resisted this temptation. It is still not possible to assume that every reader of a book such as this will be familiar with the theory of Abelian categories, and to have developed such a theory first would have seriously unbalanced the book. We have tried to use categorical methods where we could. Readers who are familiar with Abelian categories will be able to adapt the results given here to the more general setting. We refer those interested to P. Gabriel's thesis.

In Chapter 1 we merely recap on the basic notions of modules. In Chapter 2 we introduce injective modules and construct the injective envelope of a module. Most of the results of this chapter will be common to any book which does more than just mention injective modules. This chapter ends with a description of the indecomposable injective modules over a commutative Noetherian ring in terms of the prime ideals of the ring. This is in preparation for Chapter 4.

Chapter 3 is concerned with semi-simplicity. The various results in Chapters 4 and 5 on the duality that exists between Noetherian and Artinian modules have their origins in this chapter. Chapter 4 brings us to the heart of the material – the Noetherian theory. Here we deal with primary decomposition and with a characterization of Artinian modules over Noetherian rings. In Chapter 5 we show how the ring of endomorphisms of an indecomposable injective module may be used to treat localization of rings and completion of local rings. We also develop a theory of duality for complete local rings. In Chapter 6 we use the theory of injective modules to obtain complete sets of invariants for direct sum decompositions of modules.

We have added some exercises at the end of each chapter. Some of these are routine and others extend the results of the text. Also, we have included brief notes on each chapter. We have not given credit for the various results as they arise in the text; instead, we have tried to do this in the notes. We are sorry if we have failed to give credit where credit is due. We have also included a short bibliography at the end of the book. This is not intended to be complete, but rather to set down some of our sources and to suggest further reading for those who remain dissatisfied by what is offered here.

There are other books which deal with injective modules, notably the lecture notes of C. Faith and the book by J. Lambek. These two are basically concerned with the applications of injective modules to ring theory, i.e. to the construction of rings of quotients and related topics. These we have hardly touched upon. Thus there is only the obvious intersection between this book and their books.

This book arose out of a seminar given in the University of Sheffield during the session 1966/67, and we are happy to express our gratitude to the members of our audience for their patience and for their many helpful suggestions. Originally, we had intended to give only a few lectures, but Professor D. G. Northcott urged us to continue. He kindly agreed to read our manuscript and made innumerable suggestions which went well beyond routine improvements. Chapter 5 in particular was completely rewritten at his suggestion. He has been a constant encouragement to us when we have grown dispirited, and we gladly acknowledge the debt that we owe to him. At the same time, we accept full responsibility for any deficiencies that might remain.

We are also indebted to the Cambridge University Press for their patience towards us and for their unfailing helpfulness; and in particular we express our thanks to Professor C. T. C. Wall, one of the editors of this series.

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Some remarks to the intending reader

One difficulty in writing a tract of this nature, where space is confined and the scope is limited, is knowing where to begin. Our subject is injective modules, but the emphasis is set squarely on the word 'injective', so much so that we must assume the reader to be familiar already with the notion of a module. This rather implies that he will have encountered rings. *Unless we state explicitly, we shall not assume that our rings are commutative, but we shall always assume that every ring has an identity element.* The symbol R will be used consistently to denote a ring without further comment. The identity element of R will be denoted by 1_R, or less explicitly by 1, and its zero element by 0_R, or just 0. We do *not* assume in general that $1_R \neq 0_R$. If $1_R = 0_R$, then the ring possesses the single element 0_R and is called a *trivial ring*; if $1_R \neq 0_R$, then R is said to be *non-trivial*.

We assume that the reader has encountered *ideals* of rings; there are three sorts, left ideals, right ideals and two-sided ideals. When I is a two-sided ideal of R, the *residue class ring* R/I can be formed; we assume that the reader knows how this can be done.

By a *module*, we shall always mean a *unitary left module*. Thus, if M is an R-module, then M is an additive Abelian group which is such that, given $r \in R$ and $m \in M$, there is defined an element rm in M; further

$$(r_1+r_2)\,m = r_1m+r_2m, \quad r(m_1+m_2) = rm_1+rm_2,$$
$$r_1(r_2m) = (r_1r_2)\,m, \qquad 1_Rm = m$$

for all $r, r_1, r_2 \in R$ and all $m, m_1, m_2 \in M$. We shall assume that the reader is familiar with the concepts of *homomorphism* and *isomorphism* for modules, and with *submodules* and *factor modules*.

It is at this point that, rather tentatively, we take up the story, selecting only those aspects of module theory with which we are especially concerned. We are really expecting that much of the material in Chapter 1 will be familiar to the reader, and invite him to sample it to see if this is indeed the case. Those who need rather more background material than we have provided are referred, for example, to D. G. Northcott *Lessons on Rings, Modules and Multiplicities* (Cambridge, 1968)† Chapter 1, where the basic notions are introduced in a leisurely fashion.

We issue a further warning. There will be a number of occasions when Zorn's Lemma is used, and the reader should be prepared for these. A short account of this is to be found in Section 2.1 of *DGN Lessons*.

There is one matter of terminology which may cause confusion unless the reader is prepared for it. A mapping $f: A \to B$, where A and B are non-empty sets, is said to be *injective* if $f(a_1) \neq f(a_2)$ whenever a_1 and a_2 are distinct elements of A. This should not be confused with the concept of an injective module.

† In the sequel, we shall refer to this book as *DGN Lessons*.