1. *Preliminaries*

1.1 A set-theoretical result

One of our primary tasks will be to embed a given R-module in an injective module. This will be done in Chapter 2. For many purposes, we actually want to be able to say that every module is a submodule of an injective module. This can be deduced from the result on embedding by means of a device which is essentially of a set-theoretic rather than an algebraic nature. We shall deal with this point in our first result.

PROPOSITION 1.1 *Let $f: E \to E'$ be an embedding† of the R-module E in the R-module E'. Then there is an extension module E'' of E and an isomorphism $g: E'' \to E'$ which extends the embedding f, i.e. which is such that $g(e) = f(e)$ for every $e \in E$.*

Proof. We first look at the set $E' \backslash f(E)$ of all elements of E' which do not belong to $f(E)$. It may happen that this set has elements in common with E, so we consider instead any set X which is an identical copy of $E' \backslash f(E)$ but is disjoint from E. There will be a bijection $\nu: X \to E' \backslash f(E)$. Put $E'' = E \cup X$. The mapping $g: E'' \to E'$ is now defined by

$$g(e'') = \begin{cases} f(e'') & \text{if} \quad e'' \in E \\ \nu(e'') & \text{if} \quad e'' \in X. \end{cases}$$

Then g is a bijection and extends f.

We now use g to give E'' the structure of an R-module in such a way that E is a submodule of E'' and g is an R-homomorphism. Let $e_1, e_2 \in E''$ and $r \in R$. Then $g(e_1), g(e_2) \in E'$, so that $g(e_1) + g(e_2)$ and $rg(e_1)$ are defined in E'. We define $e_1 + e_2$ and re_1 in E'' by

$$e_1 + e_2 = g^{-1}(g(e_1) + g(e_2)), \quad re_1 = g^{-1}(rg(e_1)). \qquad (1.1.1)$$

These definitions agree with the addition and multiplication by

† The word 'embedding' is just another name for a monomorphism, i.e. a homomorphism which is an injective mapping.

ring elements on E, and they give E'' the structure of an R-module. It will thus be an extension module of E. Further, from (1.1.1),

$$g(e_1+e_2) = g(e_1)+g(e_2), \quad g(re_1) = rg(e_1),$$

so that g is an R-homomorphism. □

1.2 Sums and intersections of submodules

Suppose that we have an R-module M and a family $\{M_i\}_{i\in I}$ of submodules of M. The *sum*, $\sum_{i\in I} M_i$, of this family is the set of all elements $\sum_{i\in I} m_i$, where $m_i \in M_i$ for each $i \in I$ and $m_i = 0$ for all but a finite number of i. To include the case when I is the empty set, we define $\sum_{i\in I} m_i$ when I is empty to be 0. Then $\sum_{i\in I} M_i$ is a submodule of M; it is the smallest submodule of M to contain every M_i. When I is the empty set, $\sum_{i\in I} M_i$ is the zero submodule of M. The largest submodule of M which is contained in every M_i is the intersection $\bigcap_{i\in I} M_i$ of the family. The appropriate convention for this intersection when I is empty is that it is M. If I is a finite non-empty set, say $I = \{1, 2, \ldots, n\}$, then the sum and intersection are also written

$$M_1+M_2+\ldots+M_n \quad \text{and} \quad M_1 \cap M_2 \cap \ldots \cap M_n$$

respectively.

PROPOSITION 1.2 (The modular law) *Let H, K, L be submodules of an R-module M, and suppose that $K \subseteq H$. Then*

$$H \cap (K+L) = K+(H \cap L).$$

Proof. Clearly $K+(H\cap L) \subseteq H \cap (K+L)$. Consider an element h of $H \cap (K+L)$. Then $h = k+l$ for some $k \in K$, $l \in L$, and $l = h-k \in H$. Thus h belongs to $K+(H\cap L)$. This shows that $H \cap (K+L) \subseteq K+(H\cap L)$. □

In technical terms, Proposition 1.2 says that the submodules of a given module form a modular lattice with respect to the operations of addition and intersection.

Now consider a subset G of the R-module M. The intersection of all submodules of M containing G will be the smallest submodule of M to contain G; it is called the submodule of M *generated* by G (or the submodule of M generated by the *elements* of G). If G is the empty set, this is just the zero submodule of M; otherwise it consists of all elements of the form

$$r_1 g_1 + r_2 g_2 + \dots + r_n g_n,$$

where the $r_i \in R$ and the $g_i \in G$. We denote this submodule by RG.

A submodule of M which can be generated by a finite set of elements is said to be *finitely generated*; a submodule which can be generated by a single element is said to be *singly generated* or *cyclic*. For example, the ring R as an R-module is singly generated by its identity element. If $m_1, m_2, \dots, m_n$ belong to M, then the submodule that these elements generate consists of all elements of the form

$$r_1 m_1 + r_2 m_2 + \dots + r_n m_n,$$

where the $r_i \in R$; it is denoted by

$$Rm_1 + Rm_2 + \dots + Rm_n.$$

If $\{M_i\}_{i \in I}$ is a family of submodules of M and if, for each i, M_i is generated by the set of elements G_i, then $\sum_{i \in I} M_i$ is generated by the union $\bigcup_{i \in I} G_i$ of the G_i. In particular, $\sum_{i \in I} M_i$ is generated by $\bigcup_{i \in I} M_i$.

A module is said to be *simple* if (i) it is non-zero and (ii) the only proper† submodule that it possesses is the zero submodule. If M is a simple R-module and if m is any non-zero element of M, then the submodule generated by m must be M itself. Thus we have the next result.

PROPOSITION 1.3 *Every simple module is singly generated.* □

Let K be a submodule of M and let A be a left ideal of R. We denote by AK the submodule of M generated by all elements of the form ak, where $a \in A$ and $k \in K$. In fact, AK is the set of all elements of the form

$$a_1 k_1 + a_2 k_2 + \dots + a_n k_n,$$

where the $a_i \in A$ and the $k_i \in K$.

† The *proper* submodules of an R-module M are the submodules other than M itself.

Cambridge University Press
978-0-521-09092-6 - Injective Modules
D. W. Sharpe and P. Vamos
Excerpt
More information

1.3 Direct sums and direct products

Let $\{E_i\}_{i \in I}$ be a family of R-modules. Suppose initially that I is not empty, and consider the set E of all families $\{e_i\}$, where $e_i \in E_i$. This set can be given the structure of an R-module: we define

$$\{e_i\} + \{e'_i\} = \{e_i + e'_i\},$$

$$r\{e_i\} = \{re_i\},$$

where $r \in R$ and $e_i, e'_i \in E_i$. We call E the *direct product* of the family $\{E_i\}_{i \in I}$ and denote it by

$$\prod_{i \in I} E_i.$$

The direct product of an empty family of R-modules is defined to be a zero module.

Let E' be the subset of E consisting of all families $\{e_i\}$ for which $e_i = 0$ for all but a finite number of i. Then E' is a submodule of E. We call E' the *external direct sum* of the family $\{E_i\}_{i \in I}$ and denote it by

$$\bigoplus_{i \in I} E_i.$$

Of course, if I is a finite set then the external direct sum and the direct product coincide. If $I = \{1, 2, \ldots, n\}$, we may then denote the external direct sum by

$$E_1 \oplus E_2 \oplus \ldots \oplus E_n.$$

Let M be an R-module and let G be a generating set for M. For example, G could be the whole of M. We consider the family of R-modules indexed by G, each of the modules of the family being R itself. We denote the external direct sum of this family by $\bigoplus_{g \in G} R$. We can define a mapping

$$\phi\colon \bigoplus_{g \in G} R \to M$$

by

$$\phi(\{r_g\}) = \sum_{g \in G} r_g g \quad (r_g \in R).$$

Note that $r_g = 0$ for all but a finite number of g, so that ϕ is well-defined. In fact, ϕ is an R-homomorphism and also a surjective mapping, or what we shall call an *epimorphism*. We may state this as follows:

PROPOSITION 1.4 *Every R-module is a homomorphic image of an external direct sum of copies of R. If the R-module is finitely generated, then it is a homomorphic image of an external direct sum of a finite number of copies of R.* □

Let M be an R-module and let $\{M_i\}_{i\in I}$ be a family of submodules of M. We say that *the sum* $\sum_{i\in I} M_i$ *is direct*, or that it is an *internal direct sum*, and we write it as

$$\sum_{i\in I} M_i \quad \text{(d.s.)},$$

if every element of $\sum_{i\in I} M_i$ has a *unique* representation in the form $\sum_{i\in I} m_i$, where $m_i \in M_i$ and $m_i = 0$ for all but a finite number of i. The sum of an empty family of submodules is direct.

PROPOSITION 1.5 *Let $\{M_i\}_{i\in I}$ be a family of submodules of an R-module M. Then the following statements are equivalent:*

(*a*) $M = \sum_{i\in I} M_i \quad$ (d.s.);

(*b*) $M = \sum_{i\in I} M_i$ *and, for each* $j \in I$, $M_j \cap (\sum_{i\neq j} M_i) = 0$.

Proof. Assume (*a*). Then certainly $M = \sum_{i\in I} M_i$. Now consider an element j of I and an element m_j of $M_j \cap (\sum_{i\neq j} M_i)$. Then we can write

$$m_j = \sum_{i\neq j} m_i,$$

where $m_i \in M_i$ and $m_i = 0$ for all but a finite number of i. Then

$$m_j + \sum_{i\neq j} (-m_i) = 0.$$

It follows from the uniqueness of sums that $m_j = 0$. This proves (*b*).

Now assume (*b*), and suppose

$$\sum_{i\in I} m_i = \sum_{i\in I} m'_i,$$

where $m_i, m'_i \in M_i$ and $m_i = m'_i = 0$ for all but a finite number of i. Consider an element j of I. Then

$$m_j - m'_j = \sum_{i\neq j} (m'_i - m_i),$$

so

$$m_j - m'_j \in M_j \cap (\sum_{i\neq j} M_i) = 0.$$

Hence $m_j = m'_j$, and this is true for every j in I. This establishes (a). □

Staying with the family $\{M_i\}_{i\in I}$ of submodules of M, we can form their external direct sum and can then define a mapping

$$f: \bigoplus_{i\in I} M_i \to \sum_{i\in I} M_i$$

by
$$f(\{m_i\}) = \sum_{i\in I} m_i,$$

where $m_i \in M_i$. This mapping is an epimorphism, and is an isomorphism if and only if the sum $\sum_{i\in I} M_i$ is direct. Thus any result about external direct sums can be transferred by means of the mapping f to a corresponding result about internal direct sums.

We return again to the family of R-modules $\{E_i\}_{i\in I}$ and denote by E' its external direct sum. For each $j\in I$, denote by E'_j the set of all elements of E' of the form $\{e_i\}$, where $e_i = 0$ if $i \neq j$. Then E'_j is a submodule of E' isomorphic to E_j and

$$E' = \sum_{i\in I} E'_i \quad \text{(d.s.)}.$$

Thus any result about internal direct sums can be transferred to a corresponding result about external direct sums.

We shall in future dispense with the adjectives 'external' and 'internal' and rely on the context to determine which type of direct sum is meant. We shall also feel entirely free to take results concerning one and apply their analogues for the other; and this we shall do without comment.

For each j in I, we can define mappings

$$\phi_j: E_j \to \prod_{i\in I} E_i \quad \text{and} \quad \pi_j: \prod_{i\in I} E_i \to E_j.$$

If $e_j \in E_j$, we put $\phi_j(e_j) = \{e'_i\}$, where $e'_j = e_j$ and $e'_i = 0$ if $i \neq j$; and $\pi_j(\{e_i\}) = e_j$, where $e_i \in E_i$. Then, for each j, ϕ_j and π_j are homomorphisms. Also, ϕ_j is an injection and π_j a surjection, so that ϕ_j is a monomorphism and π_j an epimorphism. We call the ϕ_j the *injection mappings* and the π_j the *projection mappings* of the direct product $\prod_{i\in I} E_i$. Note that, for each $j, k \in I$, the combined mapping

$$E_j \xrightarrow{\phi_j} \prod_{i\in I} E_i \xrightarrow{\pi_k} E_k$$

is the zero mapping if $j \neq k$ and is the identity mapping if $j = k$, i.e.

$$\pi_k \phi_j = 0 \text{ if } j \neq k \quad \text{and} \quad \pi_j \phi_j = \mathrm{id}_{E_j}. \tag{1.3.1}$$

Just as for direct products, so also for external and internal direct sums we can define injection and projection mappings. In the case of an internal direct sum, the injection mappings are just inclusions.

Suppose we have a finite family $\{E_i\}_{i=1}^m$ of R-modules, where $m \geqslant 1$, and denote by ϕ_i, π_i the injection and projection mappings of its direct sum E. It is easily seen that

$$\sum_{i=1}^{m} \phi_i \pi_i = \mathrm{id}_E. \tag{1.3.2}$$

Finally, a submodule M' of an R-module M is called a *direct summand* of M if there exists a submodule M'' of M such that

$$M = M' + M'' \quad \text{(d.s.)}.$$

1.4 Some applications of Zorn's Lemma to modules

DEFINITION *Let M be an R-module. A submodule K of M is said to be a 'maximal submodule' of M if* (i) *K is a proper submodule of M and* (ii) *there is no proper submodule of M strictly containing K.*

PROPOSITION 1.6 *Let M be a finitely generated R-module. Then every proper submodule of M is contained in a maximal submodule of M.*

Proof. Let M' be a proper submodule of M, and denote by Ω the collection of all proper submodules of M which contain M'. Then Ω is not empty, because M' belongs to Ω. We may partially order Ω by inclusion. Let Σ be a non-empty totally ordered subset of Ω, and denote by M_0 the union of all the members of Σ. Then $M_0 \supseteq M'$, and it may be verified that M_0 is a submodule of M. Note that this verification uses the fact that Σ is totally ordered. Moreover, M_0 is a proper submodule of M. For suppose otherwise, and let $\{m_1, m_2, \ldots, m_n\}$ be a set of generators of M. Then each m_i belongs to a member of Σ, so there must be a member of Σ which contains every m_i, i.e. which is the whole of M. This is just not so. Thus $M_0 \in \Omega$ and is an upper bound of Σ. It follows by Zorn's Lemma that Ω possesses a maximal member K (say), i.e. M has a maximal submodule containing M'. □

COROLLARY 1 *A non-zero finitely generated R-module possesses a maximal submodule.*

Proof. Apply Proposition 1.6 to the zero submodule. □

COROLLARY 2 *Every proper left ideal of R is contained in a maximal left ideal.*

Proof. This follows from Proposition 1.6 since R is a singly generated module over itself, when its left ideals become submodules. □

It is worth noting that the proof of Proposition 1.6 breaks down if M is not finitely generated, because we cannot then be sure that M_0 is proper. For a similar sort of reason, Zorn's Lemma will not give the existence of a 'minimal' submodule of an arbitrary non-zero module.

There will be a number of occasions when we shall need to use Zorn's Lemma in situations involving direct sums, and although the contexts will be different in each case the application of the Lemma is the same. It is convenient to deal with the situation here.

Suppose we concentrate our attention on some collection $\mathscr{A}$ of submodules of M. For example, $\mathscr{A}$ could be the collection of all simple submodules of M. We denote by Ω the collection of subsets of $\mathscr{A}$ which have the property that the sum of the submodules in the subset is direct. For example, a collection containing a single submodule from $\mathscr{A}$ would belong to Ω; but certainly the empty set belongs to Ω. Then Ω is not empty and may be partially ordered by inclusion. Let Σ be a non-empty totally ordered subset of Ω. Consider the union $\mathscr{A}'$ of all members of Σ. We wish to show that $\mathscr{A}' \in \Omega$, i.e. the sum of the submodules in $\mathscr{A}'$ is direct. Put $\mathscr{A}' = \{M_i\}_{i \in I}$, and suppose that

$$\sum_{i \in I} m_i = \sum_{i \in I} m_i',$$

where $m_i, m_i' \in M_i$ and $m_i = m_i' = 0$ for all but a finite number of i. Suppose that the values of i for which m_i and m_i' are not both zero are $i_1, i_2, \ldots, i_n$. Then

$$m_{i_1} + m_{i_2} + \ldots + m_{i_n} = m'_{i_1} + m'_{i_2} + \ldots + m'_{i_n}.$$

Now each of $M_{i_1}, M_{i_2}, \ldots, M_{i_n}$ belongs to some member of Σ and Σ is totally ordered, so there is a member of Σ which contains all of $M_{i_1}, M_{i_2}, \ldots, M_{i_n}$. This must mean that $m_{i_\alpha} = m'_{i_\alpha}$ for $\alpha = 1, 2, \ldots, n$. It follows that $m_i = m'_i$ for every value of i, so that the sum of the submodules in $\mathscr{A}'$ is direct, i.e. $\mathscr{A}' \in \Omega$. Thus Σ is bounded above, so, by Zorn's Lemma, Ω has a maximal member.

For reference, we state this conclusion in a proposition.

PROPOSITION 1.7 *Let $\mathscr{A}$ be a collection of submodules of an R-module M. Then there is a maximal collection of members of $\mathscr{A}$ whose sum is direct.* □

1.5 The isomorphism theorems

Let $f: M \to N$ be a homomorphism of R-modules, let A be a submodule of M and let B be a submodule of N. Then $f(A)$ is a submodule of N contained in $\operatorname{Im} f$, the image of f, and $f^{-1}(B)$, which consists of all elements m of M such that $f(m) \in B$, is a submodule of A containing $\operatorname{Ker} f$, the kernel of f. Also,

$$f^{-1}(f(A)) = A + \operatorname{Ker} f \quad \text{and} \quad f(f^{-1}(B)) = B \cap \operatorname{Im} f,$$

so that, if $A \supseteq \operatorname{Ker} f$ then $f^{-1}(f(A)) = A$, and if $B \subseteq \operatorname{Im} f$ then $f(f^{-1}(B)) = B$. This gives the next result.

PROPOSITION 1.8 *With the above notation, there is a one–one correspondence between the submodules of M containing* $\operatorname{Ker} f$ *and the submodules of N contained in* $\operatorname{Im} f$. *This is such that, if A $(\supseteq \operatorname{Ker} f)$ and B $(\subseteq \operatorname{Im} f)$ correspond, then $B = f(A)$ and $A = f^{-1}(B)$. This correspondence preserves inclusion.* □

The last remark of Proposition 1.8 just expresses the fact that, if A_1, A_2 are submodules of M such that $A_1 \subseteq A_2$, then

$$f(A_1) \subseteq f(A_2),$$

and if B_1, B_2 are submodules of N such that $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

Note also that, if $\{A_i\}_{i \in I}$ is a non-empty family of submodules of M, each of which contains $\operatorname{Ker} f$, then

$$f\Big(\bigcap_{i \in I} A_i\Big) = \bigcap_{i \in I} f(A_i). \qquad (1.5.1)$$

We consider again an R-module M with a submodule A. We assume that the reader is familiar with the construction of the factor module M/A. A typical element of M/A is a subset of M of the form

$$m+A = \{m+a : a \in A\},$$

where $m \in M$, and $m_1 + A = m_2 + A$ $(m_1, m_2 \in M)$ if and only if $m_1 - m_2 \in A$. The operations on M/A are given by

$$(m_1+A)+(m_2+A) = (m_1+m_2)+A$$

and
$$r(m+A) = (rm)+A,$$

where $r \in R$ and $m, m_1, m_2 \in M$. We can define a mapping

$$\phi: M \to M/A$$

by $\phi(m) = m+A$ $(m \in M)$. This mapping is an epimorphism and has kernel A; it is called the *natural mapping* of M on M/A. Note that, if B is a submodule of M, then $\phi(B) = (A+B)/A$. Thus, if $B \supseteq A$, $\phi(B) = B/A$.

We now recast Proposition 1.8 in terms of the natural mapping of M on M/A.

PROPOSITION 1.9 *Let A be a submodule of an R-module M. Then there is a one–one correspondence between the submodules of M containing A and the submodules of M/A. This is such that, if B is a submodule of M containing A, then B corresponds to B/A. This correspondence preserves inclusion.* □

We note also that, if $\{A_i\}_{i \in I}$ is a non-empty family of submodules of M containing A, then

$$(\bigcap_{i \in I} A_i)/A = \bigcap_{i \in I} (A_i/A). \qquad (1.5.2)$$

This is just (1.5.1) applied to the natural mapping.

Let $f: M \to N$ be a homomorphism of R-modules, let A be a submodule of M and B a submodule of N. Suppose further that $f(A) \subseteq B$. Then we can define a mapping

$$f^*: M/A \to N/B$$

by $f^*(m+A) = f(m)+B$ $(m \in M)$. It may be verified that this does define a mapping and that this mapping is an R-homomorphism. We refer to f^* as the mapping *induced* by f.