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Characteristic classes and the cohomology of finite groups

Characteristic classes and the cohomology of finite groups

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Introduction

If G is an arbitrary finite group (more generally a finitely presented possibly infinite group) it is an easy exercise in combinatorial topology to construct a finite 2-dimensional simplicial complex with fundamental group isomorphic to G. By attaching simplexes of dimension $\geqslant 3$ in a systematic way it is possible to embed this 2-complex in a larger complex $K(G, 1)$ without changing the fundamental group, but choosing K to have universal covering space homotopy equivalent to a point. One may then define the cohomology groups $H^k(G, \mathbb{Z})$ of the discrete group G to be the cohomology groups of the space K. This definition is independent of the topological model chosen, and may indeed be copied algebraically with the coefficients $\mathbb{Z}$ replaced by some (left) $\mathbb{Z}G$-module A. It is clear that the graded ring $\{H^k(G, \mathbb{Z}), k \geqslant 0\}$ is an important invariant of the group, indeed if the homomorphism $\varphi: G_1 \to G_2$ of finite groups induces an isomorphism $\varphi^*: H^*(G_2, \mathbb{Z}) \to H^*(G_1, \mathbb{Z})$, then the groups G_1 and G_2 are isomorphic. However although the literature contains one or two striking applications of group cohomology, for example to the construction of infinite class field towers and to the study of outer automorphisms of p-groups, its systematic use as a tool has been held up by a lack of calculations for specific groups. The aim of this book is to remedy this situation partially, by exploring the connection between complex representations and integral cohomology provided by characteristic classes. In this way we obtain a subring $\mathrm{Ch}(G)$ of $H^{\mathrm{even}}(G, \mathbb{Z})$, over which, as a consequence of the Hilbert basis theorem, the integral cohomology is finitely generated as a module. Warning: if G is elementary abelian of rank $\geqslant 3$, $\mathrm{Ch}(G)$ is properly contained in $H^{\mathrm{even}}(G, \mathbb{Z})$.

The book falls into three parts: the cohomology of discrete groups (Chapters 1–4), representations, bundles and characteristic classes

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(Chapters 5–6) and applications (Chapters 7–9). In the first part we base our treatment on that first given by J.P. Serre in *Corps locaux*, but emphasise those topics such as Frobenius reciprocity and the description of the image of the restriction homomorphism in terms of stable elements, which we exploit later. Chapter 4 is devoted to the spectral sequence of a group extension and to the use of this in calculating the integral cohomology in certain cases. The problem with an iterative application of the spectral sequence, say to p-groups, is that in general it does not collapse. However even in unfavourable circumstances the theory of characteristic classes may be used to identify universal cycles, thus simplifying some of the earlier calculations in the literature. For an algebraist this part of the book will seem relatively complete.

In Chapter 4 we summarise what we need from the theory of representations over an algebraically closed field, and from the theory of complex vector bundles. In neither case do we attempt to give more than a bare outline of proofs. In defining the Chern classes of a complex vector bundle we start from 1-dimensional or line bundles, and use the isomorphism between $\mathrm{Vect}^1_{\mathbb{C}}(X)$ and $H^2(X, \mathbb{Z})$ to define c_1. The algebraist may well prefer to start from the universal classes as dual to the homology classes carried by the Schubert varieties in the Grassmann manifold $G_{n,k}$ of n-dimensional subspaces in $\mathbb{C}^k$ (k large). In Chapter 6 we extend the theory of representations from $\mathbb{C}G$-modules over a point to families of $\mathbb{C}G$-modules indexed by the points x of a parameter space X on which G acts. Besides defining the kth Chern class of a representation ρ to be the kth, Chern class of the associated flat bundle over $BG = K(G, 1)$, we also use these classes to clarify some of the calculations in Chapter 4.

The third section on applications opens with a discussion of the symmetric group S_n. We prove that stably the subring generated by all Chern classes is actually generated by the classes $c_k(\pi_n)$ of the permutation representation. If $n \gg k$ and k is even, the order of $c_k(\pi_n)$ divides the denominator of B_k/k where B_k is the kth Bernoulli number. Here we adopt the convention that $B_k/k!$ equals the coefficient of t^k in the expansion of $e^t/(e^t - 1) + t/2$. The group S_n is extreme in the sense that, although its integral cohomology is finitely generated over $\mathrm{Ch}(S_n)$, the number of module generators increases rapidly with n, see [Mn]. One cause for this phenomenon is the existence of elementary abelian subgroups of large rank in S_n, and our second application avoids this problem by restricting the p-rank of a finite group to be at most 2.

If $rk_p(G) = 1$ the p-torsion in $H^*(G, \mathbb{Z})$ is periodic and generated by Chern classes; if $rk_p(G) = 2$ the best results are obtained for groups of

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prime power order (classified by N. Blackburn for $p \geqslant 5$). Our best theorem in this direction is that if G is a p-group which is either split metacyclic, or a central extension of $\mathbb{Z}/p^{n-2}$ by $(\mathbb{Z}/p) \times (\mathbb{Z}/p)$, then $H^{\text{even}}(G, \mathbb{Z}) = \text{Ch}(G)$. However it is clear that the method used extends to other classes of groups.

Finally we combine the theory for groups of low rank with the work of D. Quillen on linear groups over fields of finite characteristic containing sufficiently many roots of unity. This leads to a theory of characteristic classes for representations of G in the algebraically closed field $\bar{\mathbb{F}}_p$. By way of example note that our earlier methods show that

$$\text{Ch}(\text{SL}(3, \mathbb{F}_p))_{(l)} = H^{\text{even}}(\text{SL}(3, \mathbb{F}_p), \mathbb{Z})_{(l)}$$

if $l \geqslant 5$ and $l \nmid p(p-1)$. If l divides $p-1$ the l-torsion can be calculated by restriction to the subgroup D of diagonal matrices, Theorem 9.5, and if $l = p$ from the theory for groups of order p^3 (Theorem 8.6 and Appendix 2).

The book ends with a purely topological appendix, proving the Riemann–Roch theorem for group representations. This is motivated by the result in algebraic geometry – heuristically the multiple $M_k s_k$ of the Newton polynomial in the Chern classes $c_1 \ldots c_k$ arises from clearing fractions both from components of the Chern character and from the coefficients of the Todd genus. It is possible to give a purely algebraic proof of this result, with the sharper bound $\bar{M}_k =$ product of *distinct* primes dividing M_k, see [E–K2], but mathematically the more illuminating approach is that used here, which combines the modern treatment of natural maps between cohomology theories with transfer with elegant calculations by J.F. Adams.

With the exception of Appendix 1 prerequisites for reading this book are basic courses in algebraic topology, homological algebra and group theory. Chapter 5 will present no problems to the reader acquainted with J.P. Serre, *Représentations linéaires des groupes finis* [Se2] and D. Housemoller, *Fibre Bundles* [Hs].

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