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Group cohomology

Preliminaries

Let G be a discrete, not necessarily finite group. Denote by Λ the integral groupring $\mathbb{Z}G$ of G, consisting of formal sums $\sum n_i g_i (n_i \in \mathbb{Z}, g_i \in G)$ with the operations

$$\sum n_i g_i + \sum m_i g_i = \sum (n_i + m_i) g_i$$

and

$$(\sum n_i g_i)(\sum m_j g'_j) = \sum (n_i m_j)(g_i g'_j).$$

We shall be concerned with the abelian category $\mathfrak{A}_G$ of left Λ-modules and Λ-homomorphisms. The Λ-module A may be thought of as being defined by an abelian group A together with a homomorphism from G into Aut(A) – in short a G-action on A. We shall frequently refer to A as a G-module and write A_0 for the underlying $\mathbb{Z}$-module or abelian group. Denote by

$$A^G = \{a \in A : ga = a \text{ for all } g \in G\}$$

the subset of invariant elements. With these conventions we may define a G-action on $\mathrm{Hom}(A_0, A'_0)$ by the rule

$$(gf)(a) = gf(g^{-1}a),$$

from which it is clear that

$$\mathrm{Hom}_G(A, A') = \mathrm{Hom}(A_0, A'_0)^G.$$

Dually we define the diagonal G-action on $A \otimes A'$ by $g(a \otimes a') = ga \otimes ga'$. Note that both for homomorphisms and for tensor products with respect to the underlying commutative ring $\mathbb{Z}$ we omit the ring from the notation.

As in all categories of modules we have projective and injective objects; however these are too restrictive for our purposes, and their homologically significant properties are shared by coinduced and induced modules. Thus

Definition

Let X be an abelian group with trivial G-action (i.e. the image of G in $\mathrm{Aut}(A)$ is the identity). A is said to be coinduced if

$$A = \mathrm{Hom}(\Lambda, X),$$

and induced if

$$A = \Lambda \otimes X.$$

Exercise

If G is finite show that the notions of coinduced and induced modules coincide.

The technical usefulness of such modules is shown by

Lemma 1.1

Every G-module A embeds in a coinduced module.

Proof. Consider the map

$$A \to \mathrm{Hom}(\Lambda, A_0) \text{ given by } a \longmapsto f_a,$$

where $f_a(1) = a$ and f_a is extended to all of Λ by linearity. One checks easily that this map is (1–1) and is compatible with the G-actions.

Dually the tensor product $\Lambda \otimes A_0$ has the module A as homomorphic image – map $g \otimes a$ to ga. To see that this map is an epimorphism consider the splitting as abelian groups given by $a \longmapsto 1 \otimes a$.

Lemma 1.2

The map $A \to A^G$ is left exact. Thus if

$$A \underset{\phi}{\rightarrowtail} B \underset{\psi}{\twoheadrightarrow} C$$

is a short exact sequence of left Λ modules, the sequence

$$A^G \underset{\phi^G}{\rightarrowtail} B^G \underset{\psi^G}{\to} C^G$$

is exact in the category of abelian groups.

Proof. The only point which is not obvious is to show that $\operatorname{Ker}\psi^G \subseteq \operatorname{Image}\phi^G$. Let $b \in B^G$ with $\psi b = 0$. Then $b = \phi a$ for some $a \in A$. However

$$\phi(ga) = g(\phi a) = gb = b,$$

so ga and a have the same image under the monomorphism ϕ. It follows that a is an invariant, as required.

In general ψ^G is not an epimorphism, and this fact motivates the definition of the first cohomology group $H^1(G, A)$. Formally let us fix the group G and allow A to run through the objects of the category $\mathfrak{A}_G$.

Definition

The cohomology groups $H^k(G, A)$, $k \geqslant 0$, form a covariant family of functors from $\mathfrak{A}_G$ to abelian groups, which has the following properties:

(1) $H^0(G, A) = A^G$,

(2) For each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathfrak{A}_G$ there exists a natural transformation $\delta = \delta^k: H^k(G, C) \to H^{k+1}(G, A)$ and a long exact sequence of cohomology groups

$$\cdots \to H^k(G, A) \underset{\phi^*}{\to} H^k(G, B) \underset{\psi^*}{\to} H^k(G, C) \underset{\delta}{\to} H^{k+1}(G, A) \to \cdots.$$

(3) If A is a coinduced module, then $H^k(G, A) = 0$ for all $k \geqslant 1$.

In short the family $\{H^k(G, \cdot), k \geqslant 0\}$ is a cohomological extension of the invariant element functor, which vanishes on coinduced modules.

Theorem 1.3

The cohomological extension $\{H^k(G, \cdot), k \geqslant 0\}$ *exists and is unique.*

Proof. If $\mathbb{Z}$ is given the trivial G-structure, then $A^G = \operatorname{Hom}_G(\mathbb{Z}, A)$. This is so since any G-homomorphism is determined by the image of $1 \in \mathbb{Z}$. Now let

$$\cdots \to P_k \to P_{k-1} \to \cdots P_0 \twoheadrightarrow \mathbb{Z}$$

be an exact sequence of modules over the group ring Λ with each module P_k projective. Such a sequence is called a projective resolution of the trivial G-module $\mathbb{Z}$, and a specific example will be given below. By a standard argument in homological algebra any two such are chain homotopy equivalent. The composition of two successive homomorphisms in the related sequence

$$\cdots \leftarrow \operatorname{Hom}_G(P_k, A) \leftarrow \operatorname{Hom}_G(P_{k-1}, A) \leftarrow \cdots \leftarrow \operatorname{Hom}_G(P_0, A) \leftarrow A^G$$

is zero for any G-module A. Thus we may define $H^k(G,A) = H_k(\mathrm{Hom}_G(P_*, A))$. The existence of the natural coboundary homomorphisms δ^k and the long exact sequence of cohomology groups (2) follow by the usual diagram chase, and property (1) is satisfied, once we make the convention that the 'boundaries' in dimension 0 are trivial.

Consider the coinduced module $\mathrm{Hom}(\Lambda, X)$ for which

$$\mathrm{Hom}_G(P_k, \mathrm{Hom}(\Lambda, X)) \cong \mathrm{Hom}(P_{k,0}, X).$$

Since the G-projective module P_k is free over $\mathbb{Z}$, applying $\mathrm{Hom}(\cdot, X)$ to the projective resolution preserves exactness. Hence

$$H^k(G, \mathrm{Hom}(\Lambda, X)) = 0 \quad \text{for} \quad k \geqslant 1.$$

Uniqueness is obvious in dimension zero. Inductively assume that we have proved this up to dimension $k-1$, and consider the short exact sequence of coefficients in $\mathfrak{A}_G$

$$A \rightarrowtail \mathrm{Hom}(\Lambda, A_0) \twoheadrightarrow \bar{A}.$$

By properties (2) and (3)

$$\delta^{k-1}: H^{k-1}(G, \bar{A}) \cong H^k(G, A)$$

is an isomorphism, so that uniqueness also holds in dimension k. Note that in dimension 1 δ is only an epimorphism, but this is sufficient.

The technique just used to extend a result or construction from dimension 0 to dimension k is called *dimension shifting*. We shall use it frequently in what follows.

In order to complete the proof of (1.3) it remains to define the standard resolution for an arbitrary discrete group. Let $\bar{P}_k$ be the free $\mathbb{Z}$-module with basis given by $(k+1)$-tuples $(g_0, \ldots, g_k)$ of elements from G, and let G act via

$$g(g_0, \ldots, g_k) = (gg_0, \ldots, gg_k).$$

As basis elements with respect to the ring Λ rather than $\mathbb{Z}$ we may take $(k+1)$-tuples with $g_0 = 1$, the identity element of G. The boundary homomorphism $\mathrm{d}: \bar{P}_{k-1}$ is defined on each free generator over $\mathbb{Z}$ by

$$\mathrm{d}(g_0, \ldots, g_k) = \sum_{j=0}^{k} (-1)^j (g_0, \ldots \hat{g}_j \ldots g_k),$$

when as usual we read $\hat{g}_j$ as 'omit the element g_j'. In dimension zero we use the augmentation map $\varepsilon: P_0 \to \mathbb{Z}$ with $\varepsilon(g_0) = 1$. Formally we have borrowed the definition of d from that of the simplicial boundary of a simplex with vertices indexed by the g_j. Hence, since this simplex is acyclic, the algebraic sequence of Λ-modules and Λ-homomorphisms

$$\cdots \rightarrow \bar{P}_k \xrightarrow[d_k]{} \bar{P}_{k-1} \xrightarrow[d_{k-1}]{} \cdots \rightarrow \bar{P}_0 \xrightarrow[\varepsilon]{} \mathbb{Z}$$

is exact, and constitutes a free resolution of $\mathbb{Z}$ over Λ. Note in passing that when G is finite, this construction shows that each module $\bar{P}_k$ may be taken to be finitely generated.

A cochain in $\mathrm{Hom}_\Lambda(\bar{P}_k, A)$ may be identified with a function

$$f: G \times \cdots \times G \rightarrow A,$$

which satisfies the equivariance condition

$$f(gg_0, \ldots, gg_k) = gf(g_0, \ldots, g_k).$$

An equivariant cochain is thus determined by restriction to elements of the form $[g_1|\cdots|g_k] = (1, g_1, g_1g_2, \ldots, g_1g_2\ldots g_k)$, from which it follows that we may interpret the elements of $\mathrm{Hom}_\Lambda(\bar{P}_k, A)$ as non-homogeneous cochains f (on k arguments), for which the coboundary d^*f is given by the formula

$$\begin{aligned} d^*f(g_1, \ldots, g_{k+1}) = g_1 f(g_2, \ldots, g_{k+1}) &- f(g_1g_2, g_3, \ldots, g_{k+1}) \\ &+ f(g_1, g_2g_3, g_4, \ldots, g_{k+1}) - \cdots \\ &+ (-1)^j f(g_1, \ldots, g_jg_{j+1}, \ldots, g_{k+1}) + \cdots \\ &+ (-1)^{k+1} f(g_1, \ldots, g_k). \end{aligned}$$

Exercise

Use this formula to check directly that $d^{*2} = 0$. Show also that it is possible to confine attention to the subcomplex of normalised cochains which satisfy the condition that $f(g_1, \ldots, g_k) = 0$ whenever some $g_j = 1$.

Although the standard resolution is important for the abstract definition of the groups $H^k(G, A)$, it is almost useless as a tool in calculations. These are best done by means of a special resolution for the group concerned, often motivated by topological considerations, and, as already noted, chain homotopy equivalent to the standard resolution above. For example let $G = C_r^T$, a cyclic group of order r generated by T, $2 \leqslant r \leqslant \infty$.

1. If $r = \infty$, one has the resolution

$$\mathbb{Z}C_\infty^S \rightarrowtail \mathbb{Z}C_\infty^T \twoheadrightarrow \mathbb{Z},$$

in which $S \rightarrow (T-1)$ and $T \rightarrow 1$.

Here the topological motivation is the elementary fact that the exponential map from the real numbers $\mathbb{R}$ to the circle S^1 is a universal covering map. The cyclic group C_∞^T acts on $\mathbb{R}$ by mapping the half-open interval $[n, n+1)$ according to the rule $Tx = (x+1)$. For an arbitrary G-module A

$$H^0(G, A) = A^G,\ H^1(G, A) = A/(T-1)A,\ H^k(G, A) = 0,\ k \geqslant 2.$$

Thus C_∞^T is an example of a group of cohomological dimension 1.

2. If $r < \infty$, write $N = 1 + T + T^2 + \cdots + T^{r-1}$, the sum of the group elements. Then by inspection the following is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}C_r^T$ – topologically we consider an equivariant cellular decomposition of a sphere with respect to an induced linear action:

$$\cdots \rightarrow \Lambda_{(2k)} \xrightarrow[N]{} \Lambda_{(2k-1)} \xrightarrow[T-1]{} \cdots \xrightarrow[N]{} \Lambda_{(1)} \xrightarrow[T-1]{} \Lambda_{(0)} \longrightarrow \mathbb{Z}.$$

For an arbitrary G-module A it follows that

$$H^0(G, A) = A^G,\ H^{2k}(G, A) = A^G/NA,\ H^{2k-1}(G, A)$$
$$= \operatorname{Ker} N/(T-1)A,\ k \geqslant 1.$$

The finite cyclic group C_r^T is an example of a group with *periodic cohomology*, a topic which we will systematically study in Chapter III below.

Low-dimensional interpretation

The formula for the coboundary shows that a 1-cocycle is a map $f: G \rightarrow A$ which satisfies the condition $f(g_1 g_2) = g_1 f(g_2) + f(g_1)$. Such a map is called a crossed homomorphism (note that when the G-action on A is trivial, a crossed homomorphism is a homomorphism in the usual sense). The coboundaries in dimension 1 are the principal crossed homomorphisms of the form $f_a(g) = ga - a$ for all $g \in G$. Thus $H^1(G, A)$ is isomorphic to the abelian group of crossed homomorphisms modulo principal crossed homomorphisms.

Similarly a 2-cocycle from the standard resolution is a map

$$f: G \times G \rightarrow A$$

which satisfies the condition

$$g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f_1(g_1, g_2) = 0.$$

Such a function is called a factor system for the following reason. Consider the family of groups E which are extensions of the abelian group A_0 by G, and for which the G-structure on A corresponds to the action of G on A_0 by conjugation. This correspondence depends on the choice of a transversal $s: G \to E$ (set of coset representatives) for A_0, in G, and such a transversal satisfies

$$s(g_1)s(g_2) = f(g_1, g_2)s(g_1 g_2).$$

By messy but straightforward calculation one then shows
(i) the factor system f determines the composition law in E,
(ii) the cocycle identity above is equivalent to associativity in E, and
(iii) choice of a new transversal $s': G \to E$ changes f by a coboundary.

Hence the second cohomology group $H^2(G, A)$ describes the family of extensions

$$A_0 \rightarrowtail E \twoheadrightarrow G$$

for a specified G-action on A. Put another way the extension groups associated with the pair (G, A), where A is abelian as a normal subgroup, are determined up to isomorphism by the module structure on A and a 2-dimensional cohomology class. For a more leisurely discussion the reader is referred to the book [Mac].

Exercises

1. Under what conditions on A is $H^1(C_\infty^T, A) = 0$?
2. Using the calculation of $H^2(C_r^T, A)$ given above, determine all possible extensions of $\mathbb{Z}$ by $\mathbb{Z}/2$, of $\mathbb{Z}/p$ by $\mathbb{Z}/2$ (p = prime), and of $\mathbb{Z}/4$ by $\mathbb{Z}/2$. (As in the general discussion we adopt the convention that the first named group corresponds to the normal subgroup in E.)

As a further illustration of extension theory consider the familiar classification of groups of order p^3, where p is an odd prime number. Such a group is either an extension of $C_{p^2}^A$ by C_p^B or of $C_p^A \times C_p^C$ by C_p^B; in both cases we must determine first the possible module structures and then the size of H^2. When the normal subgroup is cyclic, and the module structure is trivial, the possible extensions are $C_{p^2}^A \times C_p^B$ (zero element in H^2) and $C_{p^3}^B$ (generator of $H^2(C_p, \mathbb{Z}/p^2) \cong \mathbb{Z}/p$). If the module structure is non-trivial – recall that Aut (C_{p^2}) is cyclic of order $p(p-1)$ – the extension is P_1, the unique non-abelian metacyclic group of order p^3. The group H^2 is trivial, since both the invariant elements and the image of N are isomorphic to $p\mathbb{Z}/p^2\mathbb{Z}$. When the normal subgroup is non-cyclic, and the module structure is trivial, we obtain $C_p^A \times C_p^B \times C_p^C$ or $C_{p^2}^A \times C_p^B$ (counted twice). Finally, if the module structure is trivial, which implies that the generator $\bar{B}$ is mapped to a parabolic element in $GL(2, \mathbb{F}_p)$, H^2 again vanishes and the extension defines the non-abelian group P_2 of order p^3 and exponent p.

The groups of order $p^4 (p \geqslant 5)$ are listed in Appendix 3. By considering such a group as an extension of a group of order p^3 by a cyclic group of order p, it is possible to prove that this list is exhaustive. However the simple extension theory described here must be generalised to allow for a non-abelian kernel, see [Gb] or [Br, IV.6]. One of the main additional ingredients needed is the identification of the elements of order p in the group of outer automorphisms of P_1 and P_2, which correspond to genuine extensions.

Homology groups

If A is a G-module let A_G be the quotient group of A by the subgroup generated by elements of the form $ga - a$. This quotient is sometimes called the group of coinvariants of A; it is the largest quotient group of A on which G acts trivially. In a similar way to Lemma 1.2 one may show that A_G is right exact, and by arguing as in Theorem 1.3 with the complex $\{P_k \otimes_\Lambda A : k \geqslant 0\}$ one obtains the homology groups $H_k(G, A)$. Note that, since P_k and A are both given a left structure, we must first define a right structure on P_k using the rule $xg = g^{-1}x$. The homology groups are unique and satisfy the following properties:

1. $H(G, A) = A_G$,
2. For each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathfrak{A}_G$ there exists a natural transformation $\delta = \delta_k : H_k(G, C) \to H_{k-1}(G, A)$ and a long exact sequence of homology groups similar to that for cohomology.
3. If X is an abelian group, $H_k(G, \Lambda \otimes X) = 0$ for all $k \geqslant 1$, that is, the functors H_k are trivial on induced modules.

In principle the groups $H_k(G, A)$ may be calculated using the complex $\{\bar{P}_k \otimes_\Lambda A : k \geqslant 0\}$ associated to the standard resolution. An element $x \in \bar{P}_k \otimes_\Lambda A$ may be identified with the function $x(g_1, \ldots, g_k)$ taking values in A, which vanish almost everywhere. The boundary d_* is given by the formula

$$\begin{aligned} d_* x(g_1, \ldots, g_{k-1}) &= \sum_{g \in G} g^{-1} x(g, g_1, \ldots, g_{k-1}) \\ &+ \sum_{j=1}^{k-1} (-1)^j \sum_{g \in G} x(g_1, \ldots, g_j g, g^{-1}, g_{j+1}, \ldots, g_{k-1} \\ &+ (-1)^k \sum_{g \in G} x(g_1, \ldots, g_{k-1}, g). \end{aligned}$$

The most important low-dimensional interpretation is then given by

Lemma 1.4
Let $\mathbb{Z}$ have the trivial G-module structure and $[G,G]$ denote the commutator subgroup of G. Then $H_1(G,\mathbb{Z}) \cong G/[G,G]$.

Proof. As in the definition of the resolution for a cyclic group let $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$ be the augmentation homomorphism with kernel I_G equal to the subgroup of $\mathbb{Z}G$ generated by the elements $i_g = g - 1$. With this notation

$$H_0(G,A) = A/I_G A.$$

From the short exact sequence which defines I_G we see that $H_0(G,I_G) = I_G/I_G^2$ with trivial image in $H_0(G,\Lambda)$. Since the group ring Λ is certainly projective, $H_1(G,\Lambda) = 0$ and we have an isomorphism

$$\mathrm{d}_*: H_1(G,\mathbb{Z}) \xrightarrow{\simeq} H_0(G,I_G) = I_G/I_G^2.$$

The homomorphism $G \to I_G/I_G^2$ defined by $g \longmapsto i_g$ has kernel equal to $[G,G]$, from which the lemma follows.

Complete resolutions and the Tate groups

With the same notation as before multiplication by N, the sum of the group elements, defines an endomorphism $N: A \to A$ for any G-module A. Note that at this point we must restrict our attention to finite groups.

Clearly $I_G A \subseteq \operatorname{Ker} N$ and $\operatorname{Image} N \subseteq A^G$, and so N induces a homomorphism of abelian groups $N^*: H_0(G,A) \to H^0(G,A)$. Define ${}_N A$ to be the kernel of the operation of N on A, and

$$\hat{H}_0(G,A) = \operatorname{Ker} N^* = {}_N A/I_G A,$$

$$\hat{H}_0(G,A) = \operatorname{Coker} N^* = A^G/NA.$$

Lemma 1.5
If A is induced or coinduced, then $\hat{H}_0(G,A) = \hat{H}^0(G,A) = 0$.

Proof. Assume the result of the exercise following the definition of (co)induced modules, and restrict attention to $\hat{H}^0$. We may suppose that

$$A = \coprod_{g \in G} gX$$

for a suitable subgroup X of A_0. Since each $a \in X$ may be expressed uniquely as

$$a = \coprod g x_g,$$

a is an invariant if and only if all the x_g are equal, that is $a = Nx$ for some $x \in X$.

Therefore $A^G = NA$ and $\hat{H}^0(G, A) = 0$.

Lemma 1.6

If $A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence in $\mathfrak{A}_G$, *the diagram below is commutative with exact rows.*

$$\begin{array}{ccccccc} H_1(G,C) & \xrightarrow{d_*} & H_0(G,A) & \longrightarrow & (G,B) & \twoheadrightarrow & H_0(G,C) & \\ & & \downarrow N_A^* & & \downarrow N_B^* & & \downarrow N_C^* & \\ & & H^0(G,A) & \rightarrowtail & H^0(G,B) & \longrightarrow & H^0(G,C) & \xrightarrow[d^*]{} H^1(G,A) \end{array}$$

Proof. This is immediate from the definitions.

A standard argument from homological algebra (3×3 Lemma) shows that there is a connecting homomorphism $\eta: \mathrm{Ker}(N_C^*) \to \mathrm{Coker}(N_A^*)$. The same argument or an easy diagram chase shows that η may be used to splice together homology and cohomology into a long exact sequence, extending to infinity in both directions:

$$\cdots \to \hat{H}_1(G,C) \xrightarrow[d^*]{} \hat{H}_0(G,A) \to \hat{H}_0(G,B) \to \hat{H}_0(G,C) \xrightarrow[\eta]{} \hat{H}^0(G,C)$$
$$\to \hat{H}^0(G,B) \to \hat{H}^0(G,A) \xrightarrow[d^*]{} H^1(G,C) \to \cdots$$

This justifies the definition of the Tate cohomology groups $\{\hat{H}^k(G,A): -\infty < k < \infty\}$ as

$$\hat{H}^k(G,A) = H^k(G,A), \quad k \geqslant 1,$$
$$\hat{H}^0(G,A) = A^G/NA, \quad \hat{H}^{-1}(G,A) = N^A/I_G A,$$
$$\hat{H}^k(G,A) = H_{-k-1}(G,A), \quad k \leqslant -2.$$

It follows from this definition that for all G-modules A and for all $k \in \mathbb{Z}$ there are isomorphisms

$$\hat{H}^k(G,A) \cong \hat{H}^{k-1}(G, \mathrm{Hom}(\Lambda, A_0)/A) \text{ and } \hat{H}^k(G,A) \cong \hat{H}^{k+1}(G,K),$$

K is the kernel of the projection map $\Lambda \otimes A_0 \to A$ defined after Lemma 1.1.