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978-0-521-09064-3 - Miniquaternion Geometry: An Introduction to the Study of
Projective Planes

T. G. Room and P. B. Kirkpatrick

Frontmatter

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MINIQUATERNION GEOMETRY

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MINIQUATERNION GEOMETRY

AN INTRODUCTION TO THE STUDY
OF PROJECTIVE PLANES

T. G. ROOM, F.R.S.

AND

P. B. KIRKPATRICK, PH.D.

University of Sydney

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PREFACE

In the study of the elementary properties of fields, one of the difficulties, experienced perhaps more by the teacher than the taught, is that there are very few examples of explicit simple systems which resemble fields in having two group operations but do not satisfy all the conditions imposed on a field. 'Miniquaternion algebra' provides one of the simplest such systems: there are only nine elements, and the addition and multiplication tables are easy to remember, although multiplication is neither commutative nor left-distributive over addition.

From the miniquaternion algebra three different plane geometrical systems can be constructed, and the field of order nine gives rise to a fourth geometry. While these four geometries are the main subject of this book, many of the ideas developed are of much more general significance. This book is intended to provide an introduction to the relatively new subject of Projective Planes.

The authors have assumed only a knowledge of the simpler properties of groups, fields, matrices and transformations (mappings), so that the reader who has completed a first course in abstract algebra should have an adequate background for this book.

It is intended that the exercises, which vary from the very easy to the quite difficult, should form an integral part of the book.

The authors wish to record their thanks to the University of Sydney, which provided the environment in which the idea of writing this book took shape and grew, and to Westfield College in the University of London for the hospitality afforded them in the Michaelmas Term 1969 during which the final draft was written. At this time one of us (P.B.K.) received financial support from the University of Sydney as Eleanor Sophia Wood Travelling Fellow, and the other from Westfield College as a Senior Visiting Fellow.

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T. G. ROOM

P. B. KIRKPATRICK

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