INTRODUCTION

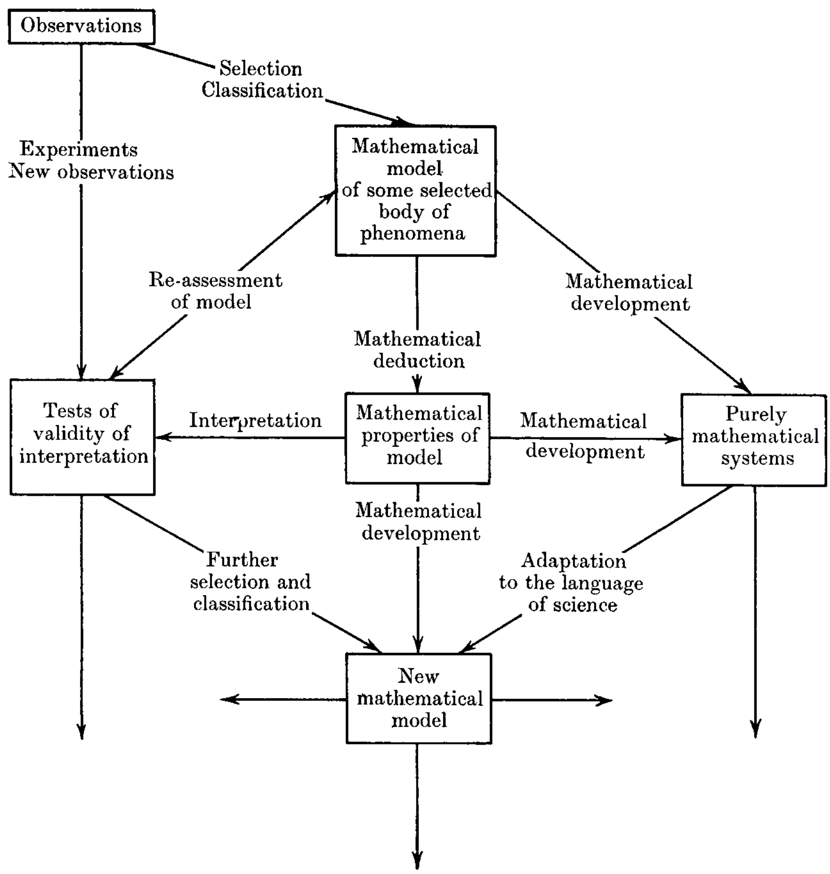
The chart on p. 2 sets out a pattern for the growth of any body of knowledge and understanding from its beginnings as a mass of uncoordinated observations ('facts') to a fully developed scientific theory. The observations first pass through the mill of classification and selection, and, because of some common characteristics, large sets of observations are grouped together as belonging to the same classes. In the next phase an interpretation of the observed relations among some of these classes is sought in terms of the mathematical relations among symbols in some mathematical system; the body of knowledge constituted by these classes of observations is represented by a system of mathematical axioms, the mathematical consequences of which should correspond to the results of other observations of phenomena within the selected field. From this stage onwards there is continual interplay among observation and experiment, selection and classification of results, and the adaptation of mathematical systems to provide mathematical counterparts of the relations among these results.

It must be appreciated that the chart refers only to what has become known today as 'fundamental research'. We may assume that it is that part that lies at a certain basic level of a three-dimensional chart representing the whole development of science and technology. From every box on this fundamental research chart there should be arrows pointing out of the plane in which it has been drawn to boxes labelled 'practical applications', these boxes being not only themselves interconnected, but also connected by arrows pointing back into various fundamental research boxes. Technology provides new tools whose influence is felt in every phase of fundamental research.

In this book we are concerned only with the very beginnings of Science, namely, the physical relationships of stationary objects on a small part of the Earth's surface, the basic Science usually called 'Geometry'. This is in no sense meant to be a historical account of the growth of the subject, rather it is an allegory. It is the story of how geometry might have developed, adapted specifically to illustrate the pattern exhibited by the chart.

The reader is to imagine himself as some primitive Farmer-Geometer; in the course of his agricultural activities he makes observations, classifies them, and then formulates rules to which all or nearly all of his observations conform. These rules are to be stated in as precise a way as possible, because they form the basic components of the 'mathematical model'; they are the 'axioms' of the 'geometry' which prescribe the relations that are to obtain among the elements and sets of elements in the geometry.

Once an axiom is formulated the reader has to change this role to that of 'Logical Machine' in which role he might be supposed to produce, indefinitely, logical consequences of the axioms, that is, the



'theorems' of the geometry. From a very few axioms quite an elaborate mathematical structure may be erected; almost any late nineteenth century text-book of geometry with its innumerable problems and riders provides an example of the less worthwhile possibilities of elaboration. On the other hand, the history of science abounds in illustrations of mathematical systems which, when devised, have seemingly no relation to physical reality but which later are found to provide the ideal mathematical model for which the physicist or chemist or biologist has been seeking.

But to return to the Farmer-Geometer; while the Logical Machine has been at work compiling lists of theorems, the Farmer-Geometer has been continuing to make observations, and against these he tests any theorem which seems capable of physical interpretation. In the light of the results of these tests he makes new classifications, and adds to or modifies the set of axioms.

At the beginning of each part of this book the formulation of the rules is discussed, and some of the more immediate and practically applicable deductions are made from them. Each step in the development of the geometry is preceded by a discussion of the physical relations from which the rules are to be derived, written in terms of the observations of the Farmer-Geometer. For simplicity of statement the scope of the main text is limited to two-dimensional geometry; the Farmer-Geometer is supposed to make his observations and conduct his experiments in a large open flat paddock or meadow surrounded by a belt of scattered trees, which thickens into impenetrable forest as the distance from the paddock increases. This forest confines his observations to the paddock and the nearest trees.

The connection between the physical observations and the mathematical model has two links. The observations of the Farmer-Geometer are necessarily gross, and his classifications are broad. All 'objects', whether trees, sticks stuck in the ground, animals grazing in the paddock, or himself as 'Observer' are treated alike as being required to be described only by their relations to each other as they appear to him, viewed along his 'lines of sight', at the instants of observation. The first link between the objects and the model is the representation on a piece of paper of the objects by 'dots' and the lines of sight by 'ink-rulings'. Thus at the first stage the physical relations among objects and lines of sight are translated into visible relations among the dots and ink-rulings of our ordinary text-book-of-Euclid diagram.

The second link is that between this diagram and the mathematical model. Without ascription to them of any inherent properties we give the name 'points' to the entities by which the objects or dots are to be represented. Among these points we prescribe relations which are meant to represent the visual relations among the dots and the ink-rulings, and, alternatively, among the objects and the lines of sight. The subject of our synthesis and investigation is the system of relations among sets of these points which is deducible from these axiomatic relations.

We wish to preserve always the distinction between the abstract 'points' and the 'objects' they represent; to emphasize this distinction, and at the same time to avoid constant repetition of phrases like 'represented by', we make the following convention: if some element of the mathematical model is denoted by say a or P, or some relation is called 'collinearity', then the physical interpretations of these will be denoted by ⁑a, ⁑P, and ⁑collinearity.

It is pertinent here to discuss at some length the nature of geometrical 'theorems', and their relation to 'diagrams' ('figures'). Essentially theorems in geometry are statements about sets of points made in the following form:

Certain relations are prescribed among points of a given set, namely, (at least in the earlier theorems) such and such sets of points are collinear. From these sets of points, possibly after the introduction of some auxiliary elements, other sets of points are 'constructed' by *naming* lines which join pairs of points, and *naming* points which are the intersections of pairs of lines. (These are the equivalents of the physical processes of 'drawing' certain lines and 'finding' their intersections.) Finally the theorem is a statement of an 'incidence relation', a relation, that is, expressible in the form 'such and such a point belongs to the collinear set determined by such and such a pair of points', which is deducible as a consequence of the given incidence relations, and the relations among the points constructed from them. The Desargues configuration (Chapter 1.3) provides the perfect example of such incidence relations.

We might therefore express a geometric theorem in the symbolic form

$$G[\{\mathrm{P}\}] \quad \text{implies} \quad T[\{\mathrm{P}\}, \{C(\mathrm{P})\}]:$$

G: the given incidence relations among the set of points $\{\mathrm{P}\}$,
$\{C(\mathrm{P})\}$: a set of points constructed from $\{\mathrm{P}\}$,
T: the deduced incidence relation among the points $\{\mathrm{P}\}$ and $\{C(\mathrm{P})\}$.

Geometry, viewed in this way, is a branch of 'Symbolic Logic'. But before the subject has developed very far it becomes clear that the format of symbolic logic is too complicated to enable proofs of even simple geometric theorems to be understood without prohibitive effort, while the invention of significant new theorems becomes practically impossible.

The geometric method consists of describing in a mixture of verbal and symbolic language certain relations among the sets of elements of a system, the description being referable to a diagram made up of dots and ink-rulings which gives a visual interpretation of these relations. Then, by logical operations, but with visual help from the diagram, further relations among the sets of elements of the system have to be deduced. These new relations are the 'theorems'.

The theorems could be stated and proved in terms which do not involve reference to any diagram: in fact the steps that are usually written out in a geometric proof do not form the complete set of steps that would be required for a full statement of the complete logical proof. But the intermediate steps, which are omitted in the geometric proof, can be supplied by the reader because of the diagram of the relations among the sets that he can see in front of him.

This makes apparent the possible weakness of the geometric method: the full logical statement must make allowance for all contingencies as each step is taken. In setting out the geometric proof we have to be careful that no unstated assumption has been made as a consequence of some apparent visual relation on the diagram, and further that some supposedly trivial specialization of the relations among the elements, for which a diagram would not make any allowance, does not in fact impose conditions that invalidate the proof of the theorem.

In the main text very few diagrams are provided, and theoretically at least the whole content could be appreciated as a rather difficult exercise in mathematical logic. But every effort has been made to set out as clearly and concisely as possible the relations among the 'given' elements and the construction of other elements from them, so that the reader may draw his own diagrams. He may check his diagrams against those at the end of the book. Moreover, but not for this reason, a definition of 'parallel lines' is introduced very early in the work, so that something very like ordinary Euclidean diagrams can be drawn. But it must be emphasized again that the mathematical model is a logical system, and diagrams are no more than a concession to human weakness; but it is a concession that the reader is strongly advised to make to himself.

The text is divided into four 'Books' marking stages in the development of the geometry. To each of these books there is added an 'Excursus' devoted in the main to the exploration of *finite* geometrical systems which illustrate the relations discussed in the text. In a finite system every point can be named and the members of every collinear set can be listed, so that the verification of a statement of geometrical relations can be reduced to the selection of appropriate combinatorial relations among subsets of a finite set.

From the point of view of the algebraically inclined mathematician the theory of these finite systems may be regarded as complete and well-known, but there is still ample scope for the amateur in the investigation of geometrical relations within particular systems. Indeed problems of current research interest lie not far beyond some of the exercises proposed on the finite geometries.

Note on the printing convention

One of the more difficult phases of the analysis of the basic concepts of mathematics is that of attaching significance to and defining 'order'. Written language consists of recognizable shapes 'ordered in space', spoken language of recognizable sounds 'ordered in time', and constructive thought would appear in effect to be a succession of mental images also 'ordered in time'. It is hard therefore to describe the concept of 'order' without relying on a natural understanding of what constitutes 'order', and, more particularly, without relying on the deeply ingrained acceptance of the printing convention of reading from left to right and down the page.

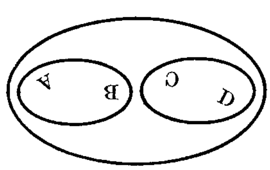
We can clarify the problem of presentation a little by supposing that the Farmer-Geometer, the writing of whose account of geometry we have taken upon ourselves, has a spoken language but no written language. We assume that he has 'written' the geometry by making marks that he invented himself, but that he makes these marks haphazardly over his writing surface. Ideally, no more than these marks would be necessary as a form of communication of his thought to others. But unless these others were equipped with high cryptographic skill they would not be likely to make much progress in understanding. We assume therefore that the Farmer-Geometer makes use of spoken language to explain the significance of the marks.

In terms of the actual writing of this book, then, we shall assume:

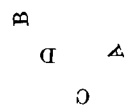
(i) That we can read the ordinary text which explains the steps that are being taken in the development of the geometry.

(ii) That in the formal presentation of the geometry we can recognise 'marks' or 'symbols', that is, when two symbols are drawn, we know whether they are the same or different. We shall in fact use as marks the letters of the alphabet and other symbols common in mathematical writing, but we use them at first only to the extent of recognizing that, for example, A and ∢ are the same, but A and B are different.

(iii) That in a set of marks we can recognize subsets; for example we can assume that we can recognise the pairs AB, CD in the set ABCD on the grounds that the original marks could have been drawn as



We shall in fact, after this discussion, usually print the symbols the 'right way up' and 'along a line', but this is only a matter of convenience and is not significant in the geometry. *We shall not assume anywhere in the text that AB is different from BA* unless some marks are added in accordance with a prescribed notation. We shall, for example, use '$\mathscr{B}|$ABC' with the clear understanding that this symbol is to convey a meaning different from that conveyed by '$\mathscr{B}|$BAC'. As a consequence, unless there is an explicit indication to the contrary, no compound symbol such as ABCD... is to be regarded as 'ordered'. It is a convenient transcription of the Farmer-Geometer's marks



In planning the book the author has had to make the choice between establishing 'order' at once on the basis of axioms which depend as little as possible on the conventions of printing, or of assuming, under the three conditions prescribed above, that printing is intelligible, and on this basis carrying the geometry forward to the stage where 'order' would most naturally be the next phase in the development. It is this latter course that has been chosen, because the concern of the book is primarily with the geometry; the argument above is resumed and 'order' is introduced formally in Chapter 1.6.

BOOK 1

GEOMETRY WITHOUT NUMBERS

CHAPTER 1.1

THE SET OF POINTS: COLLINEAR SUBSETS

We are to imagine ourselves in a state of primitive simplicity, and think of the Farmer-Geometer (F-G) ploughing in his paddock, endeavouring as he does so to synthesize a mathematical system which satisfactorily represents the relations he observes among the objects in the paddock, and, more particularly, among objects which follow the pattern of furrows made by his ploughing.

The first observation that the F-G makes is the practical one that if he wishes to plough a straight furrow he must guide his plough as he walks behind it so that two sticks that he has set up shall always appear to him to be the one directly behind, and more or less hidden by, the other. From this observation he derives his first classification among sets of his objects.

He names his markers ⁑A and ⁑B in such a way that, viewed from some position on the furrow, ⁑A (at the end of the furrow) lies directly behind ⁑B. He then ploughs the furrow so that from every position on it ⁑A appears to be behind ⁑B. On the way to ⁑B he marks two positions ⁑P and ⁑Q on the furrow. Looking back from ⁑B he sees that ⁑Q (say) lies directly behind ⁑P. He now goes beyond ⁑B and gets in a position where ⁑P is obscured by ⁑B. In this way (only he would have to look backwards to do it) he could extend his furrow up to ⁑A, and then beyond ⁑A by walking so that ⁑A covers ⁑B.

He describes this set of objects and markers along the furrow as ⁑collinear; he notes that this ⁑collinear set is certainly not the whole set of objects in the paddock, and further that, at least so far as the properties he is investigating at present are concerned, ⁑A and ⁑B do not play different parts from the other markers.

Let us now devise an abstract formulation which may be interpreted by objects having these properties. The basic undefined entities are ‘points’ which are taken to represent the ‘objects’ in the paddock. The objects may be made to assume different relations among themselves (i.e. they may be ‘moved about’ in the paddock) and any one of them may be replaced by the observer. The set of points corresponds to the set of objects in the various (spatial) relations that they assume.

Points will be denoted by roman capitals A, B, ..., P, ...; sometimes the same point may be designated by two different symbols, so that whenever points are ‘named’ (i.e. whenever they have symbols assigned to them) we have to specify, if we require each name to refer to a different point, that

'A is not the same as B'

or 'A, B, C, ... are distinct'

and we shall introduce a notation to convey this meaning.

There is at this stage only one relation among points; any given set of points either does or does not belong to a '*collinear set*'; either the given points are 'collinear' or they are not. While the relations we ascribe to points in collinear sets are suggested by the relations of objects along furrows, they are much less specific and can be interpreted in widely different physical diagrams, some of which are described in Excursus 1.

The properties we ascribe to collinear sets are contained in two sets of statements. For the present we formulate these statements as complete English sentences, as a preliminary to formulating them in Chapter 1.2 in fully symbolic form as the 'incidence' axioms $A\mathscr{I}$. The designation 'incidence axioms' is used since the axioms state the ways in which points and collinear sets are incident, i.e. the ways in which points belong to a collinear set and collinear sets contain common points.

The first set of statements is:

1. (i) Among the set of point there are subsets called collinear sets. Any two distinct points determine a collinear set to which each of the points belongs.

The meaning of this is that if we are 'given' three distinct points A, B, C we must be given either 'C belongs to the collinear set determined by A or B' or 'C does not belong to the collinear set determined by A and B'. The parts played by A and B in determining the set are indistinguishable. It is possible that there are no points other than A and B in the set.

1. (ii) If C belongs to the collinear set determined by the distinct points A and B, and B and C are distinct, then A belongs to the collinear set determined by B and C.

It follows of course that (provided A and C are distinct) B belongs to the collinear set determined by A and C, but it does not follow that the three collinear sets determined by pairs of the points A, B, C coincide, since, in diagrammatic form, we might have