1

Some Infinitary Combinatorics

This introductory chapter describes the basic combinatorial tools used in the proofs of most of the results contained in the present monograph.

Our treatment of this classical material is based on two (alternative) principles for regular cardinals: Argyros' ramification lemma (1.1) and the pressing-down lemma (1.3). The main combinatorial results established for regular cardinals – the Erdős–Rado theorem for quasi-disjoint families (1.4) and the Erdős–Rado arrow relations (1.5, 1.7) – are consequences of each of these principles. We need also a simple extension to singular cardinals of the Erdős–Rado theorem for quasi-disjoint families.

We note that sometimes, especially in Chapter 5, we obtain information on calibres of spaces directly from Argyros' ramification lemma (and a technique for singular cardinals due to Tsarpalias), rather than from some derived combinatorial result.

1.1 *Lemma*. (Argyros' ramification lemma.) Let $\omega \leq \beta \leq \alpha$ with β and α regular, let κ be a cardinal such that $0 < \kappa \ll \beta$, and for every $A \subset \alpha$ with $|A| = \alpha$ let $\mathscr{P}_A$ be a partition of A such that $|\mathscr{P}_A| < \beta$. Then there is a family $\{A_\eta : \eta < \kappa\}$ of subsets of α such that

$$|A_\eta| = \alpha \text{ for } \eta < \kappa,$$
$$A_{\eta+1} \in \mathscr{P}_{A_\eta} \text{ for } \eta < \kappa,$$
$$A_{\eta'} \subset A_\eta \text{ for } \eta < \eta' < \kappa, \text{ and}$$
$$\cap_{\eta<\kappa} A_\eta \neq \varnothing.$$

Proof. We define families $\{\mathscr{A}_\eta : \eta < \kappa\}$ such that

(i) $\mathscr{A}_0 = \{\alpha\}$;
(ii) $0 < |\mathscr{A}_\eta| < \beta$ for $\eta < \kappa$;
(iii) if $A, B \in \mathscr{A}_\eta$ and $A \neq B$, then $A \cap B = \varnothing$ for $\eta < \kappa$;
(iv) if $A \in \mathscr{A}_\eta$ then $|A| = \alpha$ for $\eta < \kappa$;

(v) $\mathscr{A}_{\eta+1} \subset \cup\{\mathscr{P}_A : A \in \mathscr{A}_\eta\}$ for $\eta < \kappa$; and
(vi) $|\alpha \backslash \cup \mathscr{A}_\eta| < \alpha$ for $\eta < \kappa$.

We proceed by recursion.

We define $\mathscr{A}_0$ by (i).

Next for $\eta < \kappa$ we define $\mathscr{A}_{\eta+1}$. We set

$$\mathscr{A}'_{\eta+1} = \cup\{\mathscr{P}_A : A \in \mathscr{A}_\eta\}, \text{ and}$$
$$\mathscr{A}_{\eta+1} = \{A \in \mathscr{A}'_{\eta+1} : |A| = \alpha\}.$$

We verify conditions (ii), (iii), (iv), (v), and (vi) for $\mathscr{A}_{\eta+1}$.

(ii) Let $A \in \mathscr{A}_\eta$. Since $|A| = \alpha$ and $|\mathscr{P}_A| < \beta \le \alpha$ and $A = \cup\mathscr{P}_A$ and α is regular, there is $B \in \mathscr{P}_A$ such that $|B| = \alpha$; we have $B \in \mathscr{A}_{\eta+1}$ and hence $\mathscr{A}_{\eta+1} \neq \varnothing$. Further, since $|\mathscr{A}_\eta| < \beta$ and $|\mathscr{P}_A| < \beta$ for $A \in \mathscr{A}_\eta$ and β is regular, we have $|\mathscr{A}'_{\eta+1}| < \beta$ and hence $|\mathscr{A}_{\eta+1}| < \beta$.

(iii), (iv) and (v) for $\mathscr{A}_{\eta+1}$ are clear from the definitions.

(vi) For $A \in \mathscr{A}_\eta$ we set $S_A = \cup\{B \in \mathscr{P}_A : |B| < \alpha\}$, and we set $S = \cup\{S_A : A \in \mathscr{A}_\eta\}$. Since α is regular and $|\mathscr{P}_A| < \alpha$ we have $|S_A| < \alpha$ for $A \in \mathscr{A}_\eta$; hence $|S| < \alpha$. Since

$$\alpha \backslash \cup \mathscr{A}_{\eta+1} = (\alpha \backslash \cup \mathscr{A}_\eta) \cup S,$$

we have $|\alpha \backslash \cup \mathscr{A}_{\eta+1}| < \alpha$, as required.

Now we assume that η is a limit ordinal such that $0 < \eta < \kappa$, and that $\mathscr{A}_\xi$ has been defined for $\xi < \eta$, and we define $\mathscr{A}_\eta$. We set

$$\mathscr{A}'_\eta = \{\bigcap_{\xi<\eta} A_\xi : A_\xi \in \mathscr{A}_\xi \text{ and } \bigcap_{\xi<\eta} A_\xi \neq \varnothing\}, \text{ and}$$
$$\mathscr{A}_\eta = \{A \in \mathscr{A}'_\eta : |A| = \alpha\}.$$

We verify conditions (ii), (iii), (iv), (v) and (vi) for $\mathscr{A}_\eta$.

(ii) We define $\varphi : \mathscr{A}'_\eta \to \prod_{\xi<\eta} \mathscr{A}_\xi$ by the rule

$$\varphi(\bigcap_{\xi<\eta} A_\xi) = \langle A_\xi : \xi < \eta \rangle.$$

It is clear that φ is a one-to-one function. Since $|\mathscr{A}_\xi| < \beta$ for $\xi < \eta$ and $|\eta| < \kappa$ and $\kappa \ll \beta$, we have $|\prod_{\xi<\eta} \mathscr{A}_\xi| < \beta$ by A.5(a); it follows that

$$|\mathscr{A}_\eta| \le |\mathscr{A}'_\eta| < \beta.$$

We set $S = \cup_{\xi<\eta}(\alpha \backslash \cup \mathscr{A}_\xi)$. Since $|\alpha \backslash \cup \mathscr{A}_\xi| < \alpha$ for $\xi < \eta$ and $|\eta| < \kappa < \alpha$ and α is regular, we have $|S| < \alpha$. Since $\alpha \backslash S = \cup \mathscr{A}'_\eta$ and $|\mathscr{A}'_\eta| < \alpha$, there is $A \in \mathscr{A}'_\eta$ such that $|A| = \alpha$; we have $A \in \mathscr{A}_\eta$ and hence $\mathscr{A}_\eta \neq \varnothing$.

(iii) and (iv) are clear for $\mathscr{A}_\eta$.

(vi) We have

$$\alpha \backslash \cup \mathscr{A}_\eta = \cup \{A \in \mathscr{A}'_\eta : |A| < \alpha\} \cup S.$$

Since $|\mathscr{A}'_\eta| < \beta \leq \alpha$ and $|S| < \alpha$, we have $|\alpha \backslash \cup \mathscr{A}_\eta| < \alpha$, as required.

The definition of the family $\{\mathscr{A}_\eta : \eta < \kappa\}$ is complete.

Since $|\alpha \backslash \cup \mathscr{A}_\eta| < \alpha$ for $\eta < \kappa$ and $\kappa < \alpha$ and α is regular, we have $|\cup_{\eta < \kappa} (\alpha \backslash \cup \mathscr{A}_\eta)| < \alpha$, and hence there is

$$\zeta \in \alpha \backslash \bigcup_{\eta < \kappa} (\alpha \backslash \cup \mathscr{A}_\eta).$$

For $\eta < \kappa$ there is $A_\eta \in \mathscr{A}_\eta$ such that $\zeta \in A_\eta$. It is clear that the family $\{A_\eta : \eta < \kappa\}$ satisfies the required conditions.

We apply Lemma 1.1 most frequently with $\beta = \alpha$. In Chapter 5 we will need the case

$$\beta = (\kappa^\kappa)^+ \leq \alpha \text{ with } \kappa \text{ and } \alpha \text{ regular.}$$

The definition of a weakly compact cardinal is given in Appendix A.

1.2 *Lemma*. Let α be a weakly compact cardinal, and for every $A \subset \alpha$ with $|A| = \alpha$ let $\mathscr{P}_A$ be a partition of A such that $|\mathscr{P}_A| < \alpha$. Then there is a family $\{A_\eta : \eta < \alpha\}$ of subsets of α such that

$$|A_\eta| = \alpha \text{ for } \eta < \alpha,$$
$$A_{\eta+1} \in \mathscr{P}_{A_\eta} \text{ for } \eta < \alpha, \text{ and}$$
$$A_{\eta'} \subset A_\eta \text{ for } \eta < \eta' < \alpha.$$

Proof. We define families $\{\mathscr{A}_\eta : \eta < \alpha\}$ such that

(i) $\mathscr{A}_0 = \{\alpha\}$;

(ii) $0 < |\mathscr{A}_\eta| < \alpha$ for $\eta < \alpha$;

(iii) if $A, B \in \mathscr{A}_\eta$ and $A \neq B$, then $A \cap B = \varnothing$ for $\eta < \alpha$;

(iv) if $A \in \mathscr{A}_\eta$ then $|A| = \alpha$ for $\eta < \alpha$; and

(v) $\mathscr{A}_{\eta+1} \subset \cup \{\mathscr{P}_A : A \in \mathscr{A}_\eta\}$ for $\eta < \alpha$.

(The argument is essentially that of Lemma 1.1. To verify (ii) for limit ordinals η such that $0 < \eta < \alpha$ we set $\gamma = \sup\{|\mathscr{A}_\xi| : \xi < \eta\}$ and we note that since $\gamma < \alpha$ we have

$$|\mathscr{A}_\eta| \leq |\mathscr{A}'_\eta| \leq \prod_{\xi < \eta} |\mathscr{A}_\xi| \leq \gamma^{|\eta|} \leq 2^\gamma \cdot 2^{|\eta|} < \alpha.)$$

Now we set $\mathscr{A} = \cup_{\eta < \alpha} \mathscr{A}_\eta$ and we define a partial order $\preccurlyeq$ on $\mathscr{A}$

by $A \preccurlyeq B$ if $A \supset B$. Then $\langle \mathscr{A}, \preccurlyeq \rangle$ is a tree of height α with $|\mathscr{A}| < \alpha$ for $\eta < \alpha$; hence there is a branch

$$\Sigma = \{A_\eta : \eta < \alpha\}$$

of $\mathscr{A}$ with $A_\eta \in \mathscr{A}_\eta$ for $\eta < \alpha$. It is clear that the family $\{A_\eta : \eta < \alpha\}$ is as required.

1.3 *Lemma.* (The pressing-down lemma.) Let $\omega \leq \kappa < \alpha$ with α and κ regular, let

$$S = \{\xi < \alpha : \mathrm{cf}(\xi) \geq \kappa\},$$

and let f be a function from S to α such that $f(\xi) < \xi$ for $\xi \in S$. Then there are $T \subset S$ with $|T| = \alpha$ and $\bar{\zeta} < \alpha$ such that $f(\xi) < \bar{\zeta}$ for all $\xi \in T$.

Proof. We suppose the lemma fails. Then for $\zeta < \alpha$ we have $|f^{-1}(\zeta)| < \alpha$ and hence there is $g(\zeta)$ such that

$$\zeta < g(\zeta) < \alpha, \text{ and}$$

$$f(\xi) \geq \zeta \quad \text{for } \xi \in S, \xi \geq g(\zeta).$$

We define $\{\zeta(\eta) : \eta < \kappa\}$ by the rule

$\zeta(0) = 0,$
$\zeta(\eta) = \sup\{\zeta(\eta') : \eta' < \eta\}$ for non-zero limit ordinals $\eta < \kappa$, and
$\zeta(\eta + 1) = g(\zeta(\eta))$ for $\eta < \kappa$;

and we set $\bar{\xi} = \sup_{\eta < \kappa} \zeta(\eta)$. The function $\eta \to \zeta(\eta)$ is an ordered-set isomorphism of κ into $\bar{\xi}$, and since κ is a regular cardinal we have $\mathrm{cf}(\bar{\xi}) = \kappa$ and hence $\bar{\xi} \in S$. For $\eta < \kappa$ we have

$$g(\zeta(\eta)) = \zeta(\eta + 1) < \bar{\xi},$$

and hence $f(\bar{\xi}) \geq \zeta(\eta)$. From $\bar{\xi} = \sup_{\eta < \kappa} \zeta(\eta)$ it then follows that $f(\bar{\xi}) \geq \bar{\xi}$, a contradiction.

The proof is complete.

We remark, retaining the notation of Lemma 1.3, that since $T = \cup_{\zeta < \bar{\zeta}} (f^{-1}(\{\zeta\}) \cap T)$ and α is regular, there are $T' \subset T$ with $|T'| = \alpha$ and $\zeta < \bar{\zeta}$ such that $f(\xi) = \zeta$ for all $\xi \in T'$.

Definition. An indexed family $\{S_i : i \in I\}$ of sets is a *quasi-disjoint* family if

$$\bigcap_{i\in I} S_i = S_j \cap S_{j'} \text{ whenever } j, j' \in I, j \neq j'.$$

It is clear that a family $\{S_i : i \in I\}$ is quasi-disjoint if and only if there is a set S such that

$$S = S_j \cap S_{j'} \text{ whenever } j, j' \in I, j \neq j'.$$

1.4 *Theorem.* Let $\omega \leq \kappa \ll \alpha$ with α regular and let $\{S_\xi : \xi < \alpha\}$ be a family of sets such that $|S_\xi| < \kappa$ for $\xi < \alpha$. Then there are $A \subset \alpha$ with $|A| = \alpha$ and a set J such that

$$S_\xi \cap S_{\xi'} = J \text{ for } \xi, \xi' \in A, \xi \neq \xi'.$$

Proof. (In the terminology of the definition above we are to show that there is $A \in [\alpha]^\alpha$ such that $\{S_\xi : \xi \in A\}$ is a quasi-disjoint family. We give two proofs.)

First Proof (using Lemma 1.1). We suppose that if $A \subset \alpha$ and $\{S_\xi : \xi \in A\}$ is quasi-disjoint then $|A| < \alpha$; and for every $A \subset \alpha$ with $|A| = \alpha$ we set

$$J_A = \bigcap_{\xi \in A} S_\xi, \text{ and}$$
$$\mathscr{B}_A = \{B \subset A : \text{if } \xi, \xi' \in B, \xi \neq \xi', \text{ then } S_\xi \cap S_{\xi'} = J_A\}.$$

Since the set $\mathscr{B}_A$ partially ordered by inclusion is inductive, there is a maximal element $B_A \in \mathscr{B}_A$. We have $|B_A| < \alpha$, and since $\{\xi\} \in \mathscr{B}_A$ for all $\xi \in A$ we have $B_A \neq \varnothing$.

For $\xi \in A \backslash B_A$ it follows from the maximality of B_A that there is $\zeta(\xi) \in B_A$ such that

$$S_\xi \cap S_{\zeta(\xi)} \supsetneqq J_A.$$

We define

$$\varphi_A : A \backslash B_A \to \cup \{\mathscr{P}(S_\zeta) : \zeta \in B_A\}$$

by the rule

$$\varphi_A(\xi) = S_\xi \cap S_{\zeta(\xi)},$$

and we set

$$\mathscr{P}_A = \{B_A\} \cup \{\varphi_A^{-1}(\{S\}) : S \in \cup \{\mathscr{P}(S_\zeta) : \zeta \in B_A\}\}.$$

Then $\mathscr{P}_A$ is a partition of A, and since $|S_\zeta| < \kappa$ for $\zeta \in B_A$ and $\kappa \ll \alpha$, we have $|\mathscr{P}(S_\zeta)| < \alpha$ and hence (since $|B_A| < \alpha$ and α is regular) we have $|\mathscr{P}_A| < \alpha$. It follows from (the case $\beta = \alpha$ of) Lemma 1.1 that there is a family $\{A_\eta : \eta < \kappa\}$ of subsets of α such that

$$|A_\eta| = \alpha \quad \text{for } \eta < \kappa,$$
$$A_{\eta+1} \in \mathscr{P}_{A_\eta} \quad \text{for } \eta < \kappa,$$
$$A_{\eta'} \subset A_\eta \quad \text{for } \eta < \eta' < \kappa, \text{ and}$$
$$\bigcap_{\eta < \kappa} A_\eta \neq \varnothing.$$

For $\eta < \kappa$ there is $S(\eta) \in \cup\{\mathscr{P}(S_\zeta) : \zeta \in B_{A_\eta}\}$ such that

$$A_{\eta+1} = \varphi_{A_\eta}^{-1}(\{S(\eta)\}) \in \mathscr{P}_{A_\eta} \quad \text{and}$$
$$\varphi_{A_\eta}(\xi) = S_\xi \cap S_{\zeta(\xi)} = S(\eta) \supsetneqq J_{A_\eta} \quad \text{for } \xi \in A_{\eta+1}.$$

Since $S(\eta) = \varphi_{A_\eta}(\xi) \subsetneqq S_\xi$ for $\xi \in A_{\eta+1}$, we have

$$J_{A_\eta} \subsetneqq S(\eta) \subset \bigcap_{\xi \in A_{\eta+1}} S_\xi = J_{A_{\eta+1}}$$

and hence $J_{A_{\eta+1}} \backslash J_{A_\eta} \neq \varnothing$ for $\eta < \kappa$; it follows that

$$\left| \bigcup_{\eta < \kappa} J_{A_\eta} \right| \geq \sum_{\eta < \kappa} |J_{A_{\eta+1}} \backslash J_{A_\eta}| \geq \kappa.$$

Now let $\xi \in \cap_{\eta<\kappa} A_\eta$. Then $\xi \in A_{\eta+1}$ for $\eta < \kappa$, and from $S_\xi \supset J_{A_\eta}$ we have

$$S_\xi \supset \bigcup_{\eta < \kappa} J_{A_\eta}$$

and hence $|S_\xi| \geq \kappa$, a contradiction.

Second Proof (using Lemma 1.3). We assume without loss of generality that $S_\xi \subset \alpha$ for $\xi < \alpha$. We set

$$S = \{\xi < \alpha : \mathrm{cf}(\xi) \geq \kappa\}$$

and we define $f : S \to \alpha$ by the rule

$$f(\xi) = \sup(S_\xi \cap \xi) \quad \text{for } \xi \in S.$$

For $\xi \in S$ we have $|S_\xi| < \kappa \leq \mathrm{cf}(\xi)$ and hence $f(\xi) < \xi$. It follows from Lemma 1.3 that there are $T \subset S$ with $|T| = \alpha$ and $\bar{\zeta} < \alpha$ such that $f[T] \subset \bar{\zeta}$.

For $\xi \in T$ we have $S_\xi \cap \bar{\zeta} \in \mathscr{P}_\kappa(\bar{\zeta})$, and since $\kappa \ll \alpha$ and α is regular we have $|\mathscr{P}_\kappa(\bar{\zeta})| < \alpha$; it follows that there are $T' \subset T$ with $|T'| = \alpha$ and $J \in \mathscr{P}_\kappa(\bar{\zeta})$ such that $S_\xi \cap \bar{\zeta} = J$ for $\xi \in T'$.

We define a function $\varphi : \alpha \to T'$ as follows. We set $\varphi(0) = \min T'$, and if $\xi < \alpha$ and $\varphi(\xi')$ has been defined for all $\xi' < \xi$ we choose $\varphi(\xi) \in T'$ such that

$$\sup\{\varphi(\xi') : \xi' < \xi\} < \varphi(\xi), \text{ and}$$

$$\sup\left(\bigcup_{\xi' < \xi} S_{\varphi(\xi')}\right) < \varphi(\xi);$$

such a choice is possible because

$$\left|\bigcup_{\xi' < \xi} S_{\varphi(\xi')}\right| \leq \sum_{\xi' < \xi} |S_{\varphi(\xi')}| \leq |\xi| \cdot \kappa < \alpha$$

and T' is cofinal in α.

We set $A = \{\varphi(\xi) : \xi < \alpha\}$ and we claim that if $\xi' < \xi < \alpha$ then

$$S_{\varphi(\xi')} \cap S_{\varphi(\xi)} = J.$$

Indeed since $\varphi(\xi'), \varphi(\xi) \in T'$ we have

$$S_{\varphi(\xi')} \cap S_{\varphi(\xi)} \cap \bar{\zeta} = J \cap J = J;$$

and if $\eta \geq \bar{\zeta}$ and $\eta \in S_{\varphi(\xi)}$, then since

$$\sup(S_{\varphi(\xi)} \cap \varphi(\xi)) = f(\varphi(\xi)) < \bar{\zeta},$$

we have $\eta \geq \varphi(\xi) > \sup S_{\varphi(\xi')}$ and hence $\eta \notin S_{\varphi(\xi')}$.

The extension to singular cardinals of the Erdős–Rado theorem for quasi-disjoint families is deferred to Theorem 1.9 below.

Though it is not needed later in this work, we note a strong converse to Theorem 1.4.

Theorem. Let κ and α be cardinals with $\kappa < \alpha$ and $\alpha \geq \omega$. If for every family $\{S_\xi : \xi < \alpha\}$ of sets with $|S_\xi| < \kappa$ for $\xi < \alpha$ there is $A \subset \alpha$ such that $|A| = \alpha$ and $\{S_\xi : \xi \in A\}$ is quasi-disjoint, then $\kappa \ll \alpha$.

Proof. Suppose there are $\beta < \alpha$, $\lambda < \kappa$ such that $\beta^\lambda \geq \alpha$, and let $\{f_\xi : \xi < \alpha\}$ be a subset of β^λ with $f_\xi \neq f_{\xi'}$ for $\xi' < \xi < \alpha$. Then f_ξ is a function from λ to β, and we set

$$S_\xi = \text{graph } f_\xi = \{\langle \eta, f_\xi(\eta) \rangle : \eta < \lambda\} \quad \text{for } \xi < \alpha.$$

Since $|S_\xi| = \lambda < \kappa$, there is $A \subset \alpha$ such that $|A| = \alpha$ and $\{S_\xi : \xi \in A\}$ is quasi-disjoint. Since $|A| = \alpha > \beta$, for $\eta < \lambda$ the function from A to β defined by $\xi \to f_\xi(\eta)$ is not one-to-one and hence there are distinct elements ξ, ξ' of A and $\varphi(\eta) < \beta$ such that

$$f_\xi(\eta) = f_{\xi'}(\eta) = \varphi(\eta);$$

it follows that $f_\zeta(\eta) = \varphi(\eta)$ for all $\zeta \in A, \eta < \lambda$ and hence $\{f_\zeta : \zeta \in A\} = \{\varphi\}$. Since $f_\zeta \neq f_{\zeta'}$ for $\zeta, \zeta' \in A$ with $\zeta \neq \zeta'$ we have $|A| = 1$, a contradiction.

Notation. Let α, κ and λ be cardinals and $n < \omega$. The *arrow notation*

$$\alpha \to (\kappa)^n_\lambda$$

denotes the following partition relation: if

$$[\alpha]^n = \bigcup_{i<\lambda} P_i,$$

then there are $A \subset \alpha$ and $i < \lambda$ such that

$$|A| = \kappa, \text{ and}$$

$$[A]^n \subset P_i.$$

If $A \subset \alpha$ and $[A]^n \subset P_i$, then A is said to be P_i-*homogeneous*.

1.5 *Theorem.* Let $\omega \leq \kappa \ll \alpha$ with α regular.

(a) If κ is regular then $\alpha \to (\kappa)^2_\lambda$ for all $\lambda < \kappa$; and

(b) if κ is singular then $\alpha \to (\kappa^+)^2_\kappa$.

Proof. (a) Let $[\alpha]^2 = \cup_{i<\lambda} P_i$. We are to show that there are $i < \lambda$ and a P_i-homogeneous subset A of α such that $|A| = \kappa$. In fact we show a bit more: for each $i_0 < \lambda$, either there are $i < \lambda$ with $i \neq i_0$ and a P_i-homogeneous subset A of α such that $|A| = \kappa$, or there is a P_{i_0}-homogeneous subset A of α such that $|A| = \alpha$. Without loss of generality we assume in what follows that $i_0 = 0$, and we give two proofs.

First Proof (using Lemma 1.1). We suppose that if $A \subset \alpha$ and A is P_0-homogeneous then $|A| < \alpha$; and for every $A \subset \alpha$ with $|A| = \alpha$ we set

$$\mathscr{B}_A = \{B \subset A : [B]^2 \subset P_0\}.$$

Since the set $\mathscr{B}_A$ partially ordered by inclusion is inductive, there is a maximal element B_A of $\mathscr{B}_A$. We have $|B_A| < \alpha$, and since $\{\xi\} \in \mathscr{B}_A$ for all $\xi \in A$ we have $B_A \neq \varnothing$.

For $\xi \in A \backslash B_A$ it follows from the maximality of B_A that there is $\zeta(\xi) \in B_A$ such that $\{\xi, \zeta(\xi)\} \notin P_0$. We choose $i(\xi) \in \lambda \backslash \{0\}$ such that

$\{\xi, \zeta(\xi)\} \in P_{i(\xi)}$, and we define

$$\varphi_A : A \backslash B_A \to B_A \times (\lambda \backslash \{0\})$$

by the rule

$$\varphi_A(\xi) = (\zeta(\xi), i(\xi)),$$

and we set

$$\mathscr{P}_A = \{B_A\} \cup \{\varphi_A^{-1}(\{\zeta, i\}) : \zeta \in B_A, i \in \lambda \backslash \{0\}\}.$$

Since $\mathscr{P}_A$ is a partition of A with $|\mathscr{P}_A| < \alpha$, it follows from (the case $\beta = \alpha$ of) Lemma 1.1 that there is a family $\{A_\eta : \eta < \kappa\}$ of subsets of α such that

$$|A_\eta| = \alpha \quad \text{for } \eta < \kappa,$$
$$A_{\eta+1} \in \mathscr{P}_{A_\eta} \quad \text{for } \eta < \kappa, \text{ and}$$
$$A_{\eta'} \subset A_\eta \quad \text{for } \eta < \eta' < \kappa.$$

For $\eta < \kappa$ there are $\bar{\zeta}(\eta) \in B_{A_\eta}$ and $i(\eta) \varepsilon \lambda \backslash \{0\}$ such that

$$A_{\eta+1} = \varphi_{A_\eta}^{-1}(\{\bar{\zeta}(\eta), \bar{i}(\eta)\}).$$

Since $\lambda < \kappa$ and κ is regular, there are $I \subset \kappa$ with $|I| = \kappa$ and $\bar{i} \in \lambda \backslash \{0\}$ such that $\bar{i}(\eta) = \bar{i}$ for all $\eta \in I$. We set

$$H = \{\bar{\zeta}(\eta) : \eta \in I\}$$

and we show that

$$|H| = \kappa, \text{ and}$$
$$[H]^2 \subset P_{\bar{i}}.$$

If $\eta, \eta' \in I$ and $\eta < \eta'$, then $\bar{\zeta}(\eta) \in B_{A_\eta}$ and

$$\bar{\zeta}(\eta') \in A_{\eta+1} \subset A_\eta \backslash B_{A_\eta};$$

it follows that $\bar{\zeta}(\eta) \neq \bar{\zeta}(\eta')$. Hence $|H| = |I| = \kappa$. Further, for $\eta, \eta' \in I$ with $\eta < \eta'$ we have

$$\bar{\zeta}(\eta') \in A_{\eta+1} = \varphi_{A_\eta}^{-1}(\{\bar{\zeta}(\eta), \bar{i}(\eta)\})$$

and hence

$$\{\bar{\zeta}(\eta), \bar{\zeta}(\eta')\} \in P_{\bar{i}(\eta)} = P_{\bar{i}};$$

it follows that $[H]^2 \subset P_{\bar{i}}$, as required.

Second Proof (using Lemma 1.3). We set

$$S = \{\xi < \alpha : \text{cf}(\xi) \geq \kappa\}.$$

For $0 < i < \lambda$ and $\xi \in S$ we define a family $\{A^i_\xi(\zeta) : \zeta < \xi\}$ as follows.

$$A^i_\xi(0) = \varnothing \quad \text{if } \{0, \xi\} \notin P_i$$
$$= \{0\} \quad \text{if } \{0, \xi\} \in P_i;$$

and if $\zeta < \xi$ and $A^i_\xi(\zeta')$ has been defined for $\zeta' < \zeta$ we set

$$\tilde{A}^i_\xi(\zeta) = \bigcup_{\zeta' < \zeta} A^i_\xi(\zeta')$$

and then

$$A^i_\xi(\zeta) = \tilde{A}^i_\xi(\zeta) \qquad \text{if } \tilde{A}^i_\xi(\zeta) \cup \{\zeta, \xi\} \text{ is not } P_i\text{-homogeneous}$$
$$= \tilde{A}^i_\xi(\zeta) \cup \{\zeta\} \quad \text{if } \tilde{A}^i_\xi(\zeta) \cup \{\zeta, \xi\} \text{ is } P_i\text{-homogeneous.}$$

We set $A^i_\xi = \cup_{\zeta < \xi} A^i_\xi(\zeta)$ and we note that $\xi \notin A^i_\xi$ (since $A^i_\xi \subset \xi$) and $A^i_\xi \cup \{\xi\}$ is P_i-homogeneous.

If there are i and ξ such that $|A^i_\xi| = \kappa$ we set $A = A^i_\xi$ and the proof is complete. We assume in what follows that $|A^i_\xi| < \kappa$ for $0 < i < \lambda$ and $\xi \in S$.

We define $g : S \to \alpha$ by

$$g(\xi) = \sup\left(\bigcup_{0 < i < \lambda} A^i_\xi\right) \quad \text{for } \xi \in S.$$

Since $\lambda < \kappa \le \text{cf}(\xi)$ we have $g(\xi) < \xi$ for $\xi \in S$. It follows from Lemma 1.3 that there are $T \subset S$ with $|T| = \alpha$ and $\bar{\zeta} < \alpha$ such that $g[T] \subset \bar{\zeta}$.

For $\xi \in T$ we set $\Phi(\xi) = \langle A^i_\xi : 0 < i < \lambda \rangle$. Writing as usual $\mathscr{P}_\kappa(\bar{\zeta}) = \{B \subset \bar{\zeta} : |B| < \kappa\}$ we have $|\mathscr{P}_\kappa(\bar{\zeta})| < \alpha$ (since $\kappa \ll \alpha$ and α is regular) and hence $|\prod_{0 < i < \lambda} \mathscr{P}_\kappa(\bar{\zeta})| < \alpha$. Since $\Phi(\xi) \in \prod_{0 < i < \lambda} \mathscr{P}_\kappa(\bar{\zeta})$ for $\xi \in T$ there is $T' \subset T$ with $|T'| = \alpha$ such that if $\xi, \xi' \in T'$ then $\Phi(\xi) = \Phi(\xi')$; we show $[T']^2 \subset P_0$.

Let $\xi', \xi \in T'$ with $\xi' < \xi$ and suppose there is i such that $0 < i < \lambda$ and $\{\xi', \xi\} \in P_i$. Since $\Phi(\xi') = \Phi(\xi)$ we have $A^i_\xi = A^i_{\xi'}$; from the fact that the sets

$$A^i_\xi \cup \{\xi\} \text{ and } A^i_{\xi'} \cup \{\xi'\}$$

are P_i-homogeneous it then follows that

$$A^i_\xi \cup \{\xi', \xi\}$$

is P_i-homogeneous. We have

$$\xi' \in A^i_\xi(\xi') \subset A^i_\xi = A^i_{\xi'},$$

a contradiction.

(b) If κ is singular we have $\kappa^+ \ll \alpha$ from Theorem A.5(c), and