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D. G. Northcott
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D. G. NORTHCOTT F.R.S.
Formerly Town Trust Professor of Mathematics at the University of Sheffield

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Preface

This account of Multilinear Algebra has developed out of lectures which I gave at the University of Sheffield during the session 1981/2. In its present form it is designed for advanced undergraduates and those about to commence postgraduate studies. At this general level the only special prerequisite for reading the whole book is a familiarity with the notion of a module (over a commutative ring) and with such concepts as submodule, factor module and homomorphism.

Multilinear Algebra arises out of Linear Algebra and like its antecedent is a subject which has applications in a great many different fields. Indeed, there are so many reasons why mathematicians may need some knowledge of its concepts and results that any selection of applications is likely to disappoint as many readers as it satisfies. Furthermore, such a selection tends to upset the balance of the subject as well as adding substantially to the required background knowledge. It is my impression that young mathematicians often acquire their knowledge of Multilinear Algebra in a rather haphazard and fragmentary fashion. Here I have attempted to weld the most commonly used fragments together and to fill out the result so as to obtain a theory with an easily recognizable structure.

The book begins with the study of multilinear mappings and the tensor, exterior and symmetric powers of a module. Next, the tensor powers are fitted together to produce the tensor algebra of a module, and a similar procedure yields the exterior and symmetric algebras. Multilinear mappings and the three algebras just mentioned form the most widely used parts of the subject and, in this account, occupy the first six chapters. However, at this point we are at the threshold of a richer theory, and it is Chapter 7 that provides the climax of the book.

Chapter 7 starts with the observation that if we re-define algebras in terms of certain commutative diagrams, then we are led to a dual concept known as a *coalgebra*. Now it sometimes happens that, on the same underlying set, there exist simultaneously both an algebra-structure and a coalgebra-structure. When this happens, and provided that the two structures interact suitably, the result is called a *Hopf algebra*. It turns out that exterior and symmetric algebras are better regarded as Hopf algebras.

This approach confers further benefits. By considering linear forms on a coalgebra it is always possible to construct an associated algebra; and, since exterior and symmetric algebras have a coalgebra-structure, this construction may be applied to them. The result in the first case is the algebra of differential forms (the Grassmann algebra) and in the second case it is the algebra of differential operators.

The final chapter deals with *graded duality*. From every graded module we can construct another graded module known as its graded dual. If the components of the original graded module are free and of finite rank, then this process, when applied twice, yields a double dual that is a copy of the graded module with which we started. For similarly restricted graded algebras, coalgebras and Hopf algebras this technique gives rise to a full duality; algebras become coalgebras and *vice versa*; and Hopf algebras continue to be Hopf algebras.

Each chapter has, towards its end, a section with the title 'Comments and exercises'. The comments serve to amplify the main theory and to draw attention to points that require special attention; the exercises give the reader an opportunity to test his or her understanding of the text and a chance to become acquainted with additional results. Some exercises are marked with an asterisk. Usually these exercises are selected on the grounds of being particularly interesting or more than averagely difficult; sometimes they contain results that are used later. Where an asterisk is attached to an exercise a solution is provided in the following section. However, to prevent gaps occurring in the argument, a result contained in an exercise is not used later unless a solution has been supplied.

Once the guide-lines for the book had been settled, I found that the subject unfolded very much under its own momentum. Where I had to consult other sources, I found C. Chevalley's *Fundamental Concepts of Algebra*, even though it was written more than a quarter of a century ago, especially helpful. In particular, the account given here of Pfaffians follows closely that given by Chevalley.

Finally I wish to record my thanks to Mrs E. Benson and Mrs J. Williams of the Department of Pure Mathematics at Sheffield University. Between them they typed the whole book; and their cheerful co-operation enabled the exacting task of preparing it for the printers to proceed smoothly and without a hitch.

Sheffield, April 1983 D. G. Northcott