1

Multilinear mappings

General remarks

Throughout Chapter 1 the letter R will denote a *commutative* ring which possesses an identity element. R is called *trivial* if its zero element and its identity element are the same. Of course if R is trivial, then all its modules are null modules.

The standard notation for tensor products is introduced in Section (1.2) and from there on we allow ourselves the freedom (in certain contexts) to omit the suffix which indicates the ring over which the products are formed. More precisely when (in Chapter 1) we are dealing with tensor products of R-modules, we sometimes use $\otimes$ rather than the more explicit $\otimes_R$. This is done solely to avoid typographical complications.

1.1 Multilinear mappings

Let $M_1, M_2, \ldots, M_p$ ($p \geq 1$) and M be R-modules and let

$$\phi: M_1 \times M_2 \times \cdots \times M_p \to M \tag{1.1.1}$$

be a mapping of the cartesian product $M_1 \times M_2 \times \cdots \times M_p$ into M. We use $m_1, m_2, \ldots, m_p$ to denote typical elements of $M_1, M_2, \ldots, M_p$ respectively and r to denote a typical element of R. The mapping ϕ is called *multilinear* if

$$\phi(m_1, \ldots, m_i' + m_i'', \ldots, m_p) = \phi(m_1, \ldots, m_i', \ldots, m_p) + \phi(m_1, \ldots, m_i'', \ldots, m_p) \tag{1.1.2}$$

and

$$\phi(m_1, \ldots, rm_i, \ldots, m_p) = r\phi(m_1, \ldots, m_i, \ldots, m_p). \tag{1.1.3}$$

(Here, of course, i is unrestricted provided it lies between 1 and p.) For example, when $p = 1$ a multilinear mapping is the same as a homomorphism of R-modules.

Suppose now that (1.1.1) is a multilinear mapping. We can derive other multilinear mappings from it in the following way. Let $h: M \to N$ be a homomorphism of R-modules. Then $h \circ \phi$ is a multilinear mapping of $M_1 \times M_2 \times \cdots \times M_p$ into N. This raises the question as to whether it is possible to choose M and ϕ so that every multilinear mapping of $M_1 \times M_2 \times \cdots \times M_p$ can be obtained in this way. More precisely we pose

Problem 1. *To choose M and ϕ in such a way that given any multilinear mapping*

$$\psi: M_1 \times M_2 \times \cdots \times M_p \to N$$

there is exactly one homomorphism $h: M \to N$ (of R-modules) such that $h \circ \phi = \psi$.

This will be referred to as the *universal problem* for the multilinear mappings of $M_1 \times M_2 \times \cdots \times M_p$.

We begin by observing that if the pair (M, ϕ) solves the universal problem, then whenever we have homomorphisms $h_i: M \to N$ $(i = 1, 2)$ such that $h_1 \circ \phi = h_2 \circ \phi$, then necessarily $h_1 = h_2$. Now suppose that (M, ϕ) and (M', ϕ') both solve our universal problem. In this situation there will exist unique R-homomorphisms $\lambda: M \to M'$ and $\lambda': M' \to M$ such that $\lambda \circ \phi = \phi'$ and $\lambda' \circ \phi' = \phi$. It follows that $(\lambda' \circ \lambda) \circ \phi = \phi$ or $(\lambda' \circ \lambda) \circ \phi = i \circ \phi$, where i is the identity mapping of M. The observation at the beginning of this paragraph now shows that $\lambda' \circ \lambda = i$ and similarly $\lambda \circ \lambda'$ is the identity mapping of M'. But this means that $\lambda: M \to M'$ and $\lambda': M' \to M$ are inverse isomorphisms. Thus if we have two solutions of the universal problem, then they will be copies of each other in a very precise sense. More informally we may say that *if Problem 1 has a solution, then the solution is essentially unique.* Before we consider whether a solution always exists, we make some general observations about free modules.

From here on, until we come to the statement of Theorem 1, we shall assume that R is *non-trivial.* We recall that an R-module which possesses a *linearly independent* system of generators is called *free* and a linearly independent system of generators of a free module is called a *base.* Now let X be a set and consider homogeneous linear polynomials (with coefficients in R) in the elements of X. These form a free R-module having the elements of X as a base. This is known as the *free module generated by* X. (If X is empty, then the free module which it generates is the null module.) Any mapping of X into an R-module N has exactly one extension to an R-homomorphism of this free module into N.

We are now ready to solve Problem 1. Let $U(M_1, M_2, \ldots, M_p)$ be the free R-module generated by the cartesian product $M_1 \times M_2 \times \cdots \times M_p$. Of course, this has the set of sequences $(m_1, m_2, \ldots, m_p)$ as a base. The elements of $U(M_1, M_2, \ldots, M_p)$ that have one or other of the forms

$$(m_1, \ldots, m_i' + m_i'', \ldots, m_p) - (m_1, \ldots, m_i', \ldots, m_p) - (m_1, \ldots, m_i'', \ldots, m_p) \tag{1.1.4}$$

and

$$(m_1, \ldots, rm_i, \ldots, m_p) - r(m_1, \ldots, m_i, \ldots, m_p) \tag{1.1.5}$$

generate a submodule $V(M_1, M_2, \ldots, M_p)$ say. Put

$$M = U(M_1, M_2, \ldots, M_p)/V(M_1, M_2, \ldots, M_p) \tag{1.1.6}$$

and define

$$\phi: M_1 \times M_2 \times \cdots \times M_p \to M \tag{1.1.7}$$

so that $\phi(m_1, m_2, \ldots, m_p)$ is the natural image of $(m_1, m_2, \ldots, m_p)$, considered as an element of $U(M_1, M_2, \ldots, M_p)$, in M. Since the elements of $U(M_1, M_2, \ldots, M_p)$ described in (1.1.4) and (1.1.5) become zero in M, ϕ satisfies (1.1.2) and (1.1.3) and therefore it is multilinear. It will now be shown that M and ϕ provide a solution to our universal problem.

To this end suppose that

$$\psi: M_1 \times M_2 \times \cdots \times M_p \to N$$

is multilinear. There is an R-homomorphism

$$U(M_1, M_2, \ldots, M_p) \to N$$

in which $(m_1, m_2, \ldots, m_p)$ is mapped into $\psi(m_1, m_2, \ldots, m_p)$. Since ψ is multilinear, the homomorphism maps the elements (1.1.4) and (1.1.5) into zero and therefore it vanishes on $V(M_1, M_2, \ldots, M_p)$. Accordingly there is induced a homomorphism $h: M \to N$ which satisfies $h(\phi(m_1, m_2, \ldots, m_p)) = \psi(m_1, m_2, \ldots, m_p)$. Thus $h \circ \phi = \psi$. Finally if $h': M \to N$ is also a homomorphism such that $h' \circ \phi = \psi$, then h and h' have the same effect on every element of the form $\phi(m_1, m_2, \ldots, m_p)$. However, these elements generate M as an R-module and therefore $h = h'$. Thus (M, ϕ) solves the universal problem. We sum up our results so far.

Theorem 1. *Let $M_1, M_2, \ldots, M_p$ $(p \geq 1)$ be R-modules. Then the universal problem for multilinear mappings of $M_1 \times M_2 \times \cdots \times M_p$ has a solution. Furthermore the solution is essentially unique (in the sense explained previously).*

Corollary. *Suppose that (M, ϕ) solves the universal problem described above. Then each element of M can be expressed as a finite sum of elements of the form $\phi(m_1, m_2, \ldots, m_p)$.*

Proof. A solution to Problem 1 has just been constructed and it would be easy to check that this particular solution has the property described in the corollary. We could then utilize the result which says that any two solutions are virtually identical. However, it is more interesting to base a proof directly on the fact that (M, ϕ) meets the requirements of the universal problem. This is the method employed here.

Suppose then that M' is the R-submodule of M generated by elements of the form $\phi(m_1, m_2, \ldots, m_p)$. Also let $h_1: M \to M/M'$ be the natural homomorphism and $h_2: M \to M/M'$ the null homomorphism. Then $h_1 \circ \phi = h_2 \circ \phi$ and therefore $h_1 = h_2$. However, this implies that $M = M'$.

Let $x \in M = M'$. Then x can be expressed in the form

$$x = r\phi(m_1, m_2, \ldots, m_p) + r'\phi(m_1', m_2', \ldots, m_p') + \cdots,$$

where the sum is finite. However,

$$r\phi(m_1, m_2, \ldots, m_p) = \phi(rm_1, m_2, \ldots, m_p)$$

and similarly in the case of the other terms. The corollary follows.

1.2 The tensor notation

Once again $M_1, M_2, \ldots, M_p$, where $p \geq 1$, denote R-modules and we continue to use $m_1, m_2, \ldots, m_p$ to denote typical elements of these modules, and r to denote a typical element of R. Let the pair (M, ϕ) provide a solution to the universal problem for multilinear mappings of $M_1 \times M_2 \times \cdots \times M_p$. It is customary to write

$$M = M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_p \tag{1.2.1}$$

and to use $m_1 \otimes m_2 \otimes \cdots \otimes m_p$ to designate the element $\phi(m_1, m_2, \ldots, m_p)$ of M. When this notation is employed, $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_p$ is called the *tensor product* of $M_1, M_2, \ldots, M_p$ over R.

It will be recalled that the solution to the universal problem is unique only to the extent that any two solutions are copies of each other. Thus we can have different *models* for the tensor product. However, if we have two such models, then they are isomorphic (as modules) under an isomorphism which matches the element represented by $m_1 \otimes m_2 \otimes \cdots \otimes m_p$ in the first model with the similarly represented element in the second. On account of this we usually do not need to specify which particular model we are using.

Next, because ϕ is multilinear, the relations

$$m_1 \otimes \cdots \otimes (m_i' + m_i'') \otimes \cdots \otimes m_p$$
$$= m_1 \otimes \cdots \otimes m_i' \otimes \cdots \otimes m_p + m_1 \otimes \cdots \otimes m_i'' \otimes \cdots \otimes m_p \tag{1.2.2}$$

and

$$m_1 \otimes \cdots \otimes rm_i \otimes \cdots \otimes m_p = r(m_1 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_p) \tag{1.2.3}$$

both hold. Moreover the corollary to Theorem 1 shows that each element of $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_p$ is a finite sum of elements of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_p$. Finally the fact that the tensor product provides the solution to the universal problem for multilinear mappings may be restated as

Theorem 2. *Given an R-module N and a multilinear mapping*

$$\psi: M_1 \times M_2 \times \cdots \times M_p \to N$$

there exists a unique R-module homomorphism h, of $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_p$ *into N, such that*

$$h(m_1 \otimes m_2 \otimes \cdots \otimes m_p) = \psi(m_1, m_2, \ldots, m_p)$$

for all $m_1, m_2, \ldots, m_p$.

We interrupt the main argument to observe that when $p=1$ the universal problem can be solved by taking M to be M_1 and ϕ to be the identity mapping. This confirms that for $p=1$ the tensor product $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_p$ is just M_1 as we should naturally expect.

The reader will have noticed that the full tensor notation is rather heavy. To counteract this we shall often use a simplified version. Indeed, because in Chapter 1 we shall only be concerned with a single ring R, it will not cause confusion if we use $M_1 \otimes M_2 \otimes \cdots \otimes M_p$ in place of the typographically more cumbersome but more explicit $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_p$. Nevertheless, in the statement of theorems and other results likely to be referred to later we shall restore the subscript which identifies the relevant ring.

So much for matters of notation. Before we leave this section we shall seek to gain insight into the nature of tensor products by examining the result of forming the tensor product of a number of *free* modules.

Suppose then that, for $1 \le i \le p$, M_i is a free R-module and that B_i is a base for M_i. The first point to note is that any mapping of $B_1 \times B_2 \times \cdots \times B_p$ into an R-module N has precisely one extension to a multilinear mapping of $M_1 \times M_2 \times \cdots \times M_p$ into N. We use this observation in the proof of our next theorem.

Theorem 3. *Let* M_i $(i=1, 2, \ldots, p)$ *be a free R-module with a base* B_i. *Then* $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_p$ *is also a free R-module and it has the elements* $b_1 \otimes b_2 \otimes \cdots \otimes b_p$, *where* $b_i \in B_i$, *as a base.*

Proof. Denote by M the free R-module generated by the set $B_1 \times B_2 \times \cdots \times B_p$. Thus the sequences $(b_1, b_2, \ldots, b_p)$ form a base for M. There is then a mapping ϕ, of $B_1 \times B_2 \times \cdots \times B_p$ into M, in which $\phi(b_1, b_2, \ldots, b_p)$ is the base element $(b_1, b_2, \ldots, b_p)$. Now, as we noted above, the mapping has an extension (denoted by the same letter) to a multilinear mapping of $M_1 \times M_2 \times \cdots \times M_p$ into M. Clearly all we need to do to complete the proof is to show that (M, ϕ) solves Problem 1.

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Suppose then that we have a multilinear mapping

$$\psi: M_1 \times M_2 \times \cdots \times M_p \to N.$$

In these circumstances there exists an R-homomorphism $h: M \to N$ which maps $(b_1, b_2, \ldots, b_p)$ into $\psi(b_1, b_2, \ldots, b_p)$. Obviously $h \circ \phi = \psi$. Moreover, if $h': M \to N$ is also a homomorphism satisfying $h' \circ \phi = \psi$, then h and h' agree on the base $B_1 \times B_2 \times \cdots \times B_p$ of M and therefore $h = h'$. The proof is therefore complete.

1.3 Tensor powers of a module

Let M be an R-module and $p \geq 1$ an integer. Put

$$T_p(M) = M \otimes_R M \otimes_R \cdots \otimes_R M, \tag{1.3.1}$$

where there are p factors. The R-module $T_p(M)$ is called the p-th *tensor power* of M. These powers will later form the components of a graded algebra known as the *tensor algebra* of M. For the moment we note that $T_1(M) = M$. Also, if M is a free R-module and B is a base of M, then $T_p(M)$ is also free and it has the elements $b_1 \otimes b_2 \otimes \cdots \otimes b_p$, where $b_i \in B$, as a base. This follows from Theorem 3.

1.4 Alternating multilinear mappings

As in the last section M denotes an R-module. In this section we shall have occasion to consider products such as $M \times M \times \cdots \times M$ and $M \otimes M \otimes \cdots \otimes M$. Whenever such a product occurs it is to be understood that the number of factors is p, where $p \geq 1$.

A multilinear mapping

$$\eta: M \times M \times \cdots \times M \to N$$

is called *alternating* if $\eta(m_1, m_2, \ldots, m_p) = 0$ whenever the sequence $(m_1, m_2, \ldots, m_p)$ contains a repetition. (To clarify the position when $p = 1$ we make it explicit that all linear mappings $M \to N$ are regarded as alternating.)

Suppose that η has this property. If $1 \leq i < j \leq p$, then on expanding

$$\eta(m_1, \ldots, m_i + m_j, \ldots, m_i + m_j, \ldots, m_p) = 0,$$

where the element $m_i + m_j$ occurs in both the i-th and j-th positions, we find that

$$\eta(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_p) + \eta(m_1, \ldots, m_j, \ldots, m_i, \ldots, m_p) = 0.$$

Thus if we interchange two terms in the sequence $(m_1, m_2, \ldots, m_p)$ the effect on $\eta(m_1, m_2, \ldots, m_p)$ is to multiply it by -1. From this observation we at once obtain

Lemma 1. *Let* $\eta: M \times M \times \cdots \times M \to N$ *be an alternating multilinear mapping and let* $(i_1, i_2, \ldots, i_p)$ *be a permutation of* $(1, 2, \ldots, p)$. *Then*

$$\eta(m_{i_1}, m_{i_2}, \ldots, m_{i_p}) = \pm\eta(m_1, m_2, \ldots, m_p),$$

where the sign is plus if the permutation is even and minus if it is odd.

Another useful observation is recorded in

Lemma 2. *Let* $\eta: M \times M \times \cdots \times M \to N$ *be a multilinear mapping and suppose that* $\eta(m_1, m_2, \ldots, m_p) = 0$ *whenever* $m_i = m_{i+1}$ *for some i. Then* η *is an alternating mapping.*

Proof. The argument just before the statement of Lemma 1 shows that $\eta(m_1, m_2, \ldots, m_p)$ changes sign if we interchange two *adjacent* terms in the sequence $(m_1, m_2, \ldots, m_p)$. Now suppose that $(m_1, m_2, \ldots, m_p)$ contains a repetition. Then either two equal terms occur next to each other or this situation can be brought about by a number of adjacent interchanges. It follows that $\eta(m_1, m_2, \ldots, m_p) = 0$ so the lemma is proved.

Consider an alternating multilinear mapping η, of $M \times M \times \cdots \times M$ into an R-module N, and suppose that $h: N \to K$ is a homomorphism of R-modules. Then $h \circ \eta$ is an alternating multilinear mapping of $M \times M \times \cdots \times M$ into K. This observation leads us to pose the following universal problem.

Problem 2. *To choose N and* η *in such a way that given any alternating multilinear mapping*

$$\zeta: M \times M \times \cdots \times M \to K$$

there exists exactly one homomorphism $h: N \to K$ *(of R-modules) such that* $\zeta = h \circ \eta$.

To solve this problem we consider the tensor power $T_p(M)$ and denote by $J_p(M)$ the submodule generated by all elements $m_1 \otimes m_2 \otimes \cdots \otimes m_p$, where $(m_1, m_2, \ldots, m_p)$ contains a repetition. (It is understood that $J_1(M) = 0$.) Put $N = T_p(M)/J_p(M)$ and let η be the mapping of $M \times M \times \cdots \times M$ into N which takes $(m_1, m_2, \ldots, m_p)$ into the natural image of $m_1 \otimes m_2 \otimes \cdots \otimes m_p$ in N. Then η is an alternating multilinear mapping. If now

$$\zeta: M \times M \times \cdots \times M \to K$$

is also an alternating multilinear mapping, then, by Theorem 2, there is a homomorphism of $M \otimes M \otimes \cdots \otimes M$ into K which takes $m_1 \otimes m_2 \otimes \cdots \otimes m_p$ into $\zeta(m_1, m_2, \ldots, m_p)$. This homomorphism vanishes on $J_p(M)$ and so it induces a homomorphism $h: N \to K$. It is clear that $h \circ \eta = \zeta$. If we have a second homomorphism, say $h': N \to K$, and this satisfies $h' \circ \eta = \zeta$, then h and h' agree on the elements $\eta(m_1, m_2, \ldots, m_p)$ and therefore they agree on a system of generators of N. But this ensures that $h = h'$.

It has now been shown that the universal problem for alternating multilinear mappings of $M \times M \times \cdots \times M$ has a solution. The solution is unique in the following sense. Suppose that (N, η) and (N', η') both solve Problem 2. Then there are inverse isomorphisms $\lambda: N \to N'$ and $\lambda': N' \to N$ such that $\lambda \circ \eta = \eta'$ and $\lambda' \circ \eta' = \eta$. The situation is, in fact, almost identical with that encountered in dealing with uniqueness in the case of Problem 1.

Let us suppose that (N, η) solves the universal problem for alternating multilinear mappings of $M \times M \times \cdots \times M$. Put

$$E_p(M) = N \tag{1.4.1}$$

and

$$m_1 \wedge m_2 \wedge \cdots \wedge m_p = \eta(m_1, m_2, \ldots, m_p). \tag{1.4.2}$$

Then, because η is multilinear, we have

$$\begin{aligned} m_1 \wedge \cdots \wedge (m_i' + m_i'') \wedge \cdots \wedge m_p \\ = m_1 \wedge \cdots \wedge m_i' \wedge \cdots \wedge m_p + m_1 \wedge \cdots \wedge m_i'' \wedge \cdots \wedge m_p \end{aligned} \tag{1.4.3}$$

and

$$m_1 \wedge \cdots \wedge rm_i \wedge \cdots \wedge m_p = r(m_1 \wedge \cdots \wedge m_i \wedge \cdots \wedge m_p). \tag{1.4.4}$$

But η is also alternating. Consequently $m_1 \wedge m_2 \wedge \cdots \wedge m_p = 0$ whenever $(m_1, m_2, \ldots, m_p)$ contains a repetition and, by Lemma 1,

$$m_{i_1} \wedge m_{i_2} \wedge \cdots \wedge m_{i_p} = \pm m_1 \wedge m_2 \wedge \cdots \wedge m_p, \tag{1.4.5}$$

where the plus sign is to be used if $(i_1, i_2, \ldots, i_p)$ is an even permutation of $(1, 2, \ldots, p)$ and the minus sign if the permutation is odd.

The module $E_p(M)$ is called the p-th *exterior power* of M. As we have seen it is unique in much the same sense that $T_p(M)$ is unique. Since when $p = 1$ we can solve Problem 2 by means of M and its identity mapping, we have $E_1(M) = M$.

The defining property of the exterior power is restated in the next theorem.

Theorem 4. *Given an R-module K and an alternating multilinear mapping*

$$\zeta: M \times M \times \cdots \times M \to K$$

there exists a unique R-homomorphism $h: E_p(M) \to K$ *such that*

$$h(m_1 \wedge m_2 \wedge \cdots \wedge m_p) = \zeta(m_1, m_2, \ldots, m_p)$$

for all $m_1, m_2, \ldots, m_p$ *in* M.

Corollary. *Each element of* $E_p(M)$ *is a finite sum of elements of the form* $m_1 \wedge m_2 \wedge \cdots \wedge m_p$.

A proof can be obtained by means of a trivial modification of the argument used to establish the corollary to Theorem 1, so details will not be given.

The mapping of $M \times M \times \cdots \times M$ into $E_p(M)$ which takes $(m_1, m_2, \ldots, m_p)$ into $m_1 \wedge m_2 \wedge \cdots \wedge m_p$ is multilinear and so, by Theorem 2, there is induced a homomorphism $T_p(M) \to E_p(M)$. This is surjective because the image of $m_1 \otimes m_2 \otimes \cdots \otimes m_p$ is $m_1 \wedge m_2 \wedge \cdots \wedge m_p$. We refer to $T_p(M) \to E_p(M)$ as the *canonical homomorphism* of the tensor power onto the exterior power. Note that when $p=1$ the canonical homomorphism is the identity mapping of M if we make the identifications $T_1(M) = M = E_1(M)$.

As before we let $J_p(M)$ denote the R-submodule of $T_p(M)$ generated by all products $m_1 \otimes m_2 \otimes \cdots \otimes m_p$ in which there is a repeated factor.

Theorem 5. *The canonical homomorphism $T_p(M) \to E_p(M)$ is surjective and its kernel is $J_p(M)$. Moreover $J_p(M)$ is generated, as an R-module, by all products $m_1 \otimes m_2 \otimes \cdots \otimes m_p$, where $m_i = m_{i+1}$ for some i.*

Proof. Let $J'_p(M)$ be the submodule of $T_p(M)$ generated by all $m_1 \otimes m_2 \otimes \cdots \otimes m_p$ with $m_i = m_{i+1}$ for some i, and let $J''_p(M)$ be the kernel of the canonical homomorphism. Then $J'_p(M) \subseteq J_p(M) \subseteq J''_p(M)$. Next, by Lemma 2, the mapping

$$\theta: M \times M \times \cdots \times M \to T_p(M)/J'_p(M),$$

in which $\theta(m_1, m_2, \ldots, m_p) = m_1 \otimes m_2 \otimes \cdots \otimes m_p + J'_p(M)$, is multilinear and alternating and therefore, by Theorem 4, there is a homomorphism $\lambda: E_p(M) \to T_p(M)/J'_p(M)$ such that

$$\lambda(m_1 \wedge m_2 \wedge \cdots \wedge m_p) = m_1 \otimes m_2 \otimes \cdots \otimes m_p + J'_p(M).$$

But if λ is combined with the canonical homomorphism $T_p(M) \to E_p(M)$, then the result is the natural homomorphism

$$T_p(M) \to T_p(M)/J'_p(M).$$

Thus the natural homomorphism vanishes on $J''_p(M)$ and therefore $J''_p(M) \subseteq J'_p(M)$. Accordingly $J'_p(M) = J_p(M) = J''_p(M)$ and the theorem is proved.

It is useful to know the structure of the exterior powers of a free module. Let us assume therefore that M is a free R-module and that B is one of its bases. On the set formed by all sequences $(b_1, b_2, \ldots, b_p)$ of p *distinct* elements of B we introduce the equivalence relation in which $(b_1, b_2, \ldots, b_p)$ and $(b'_1, b'_2, \ldots, b'_p)$ are regarded as equivalent if each is a permutation of the other. From each equivalence class we now select a single representative. In the next theorem the set formed by these representatives is denoted by $I_p(B)$.

Theorem 6. *Let M be a free R-module having the set B as a base. Then $E_p(M)$ is a free R-module having the elements $b_1 \wedge b_2 \wedge \cdots \wedge b_p$ as a base, where $(b_1, b_2, \ldots, b_p)$ ranges over $I_p(B)$.*

Proof. Let N be the free R-module generated by $I_p(B)$. Define a mapping

$$\eta: B \times B \times \cdots \times B \to N$$

as follows. When $(b_1, b_2, \ldots, b_p)$ contains a repetition $\eta(b_1, b_2, \ldots, b_p)$ is to be zero. When, however, $b_1, b_2, \ldots, b_p$ are all different there is a unique permutation $(b'_1, b'_2, \ldots, b'_p)$, of $(b_1, b_2, \ldots, b_p)$, such that $(b'_1, b'_2, \ldots, b'_p)$ belongs to $I_p(B)$; in this case we put

$$\eta(b_1, b_2, \ldots, b_p) = \pm(b'_1, b'_2, \ldots, b'_p),$$

where the plus sign is taken if the permutation is even and the minus sign if it is odd. The mapping η has a unique extension (denoted by the same letter) to a multilinear mapping of $M \times M \times \cdots \times M$ into N, and the construction ensures that the extension is alternating as well as multilinear.

We claim that (N,η) *solves Problem 2.* Clearly once this has been established the theorem will follow. Suppose then that

$$\zeta: M \times M \times \cdots \times M \to K$$

is an alternating multilinear mapping. Since N is free, there is a homomorphism $h: N \to K$ such that $h(b_1, b_2, \ldots, b_p) = \zeta(b_1, b_2, \ldots, b_p)$ whenever $(b_1, b_2, \ldots, b_p)$ is in $I_p(B)$. Thus $h \circ \eta$ and ζ are alternating multilinear mappings which agree on $I_p(B)$ and therefore on $B \times B \times \cdots \times B$ as well. If follows that $h \circ \eta = \zeta$. Moreover it is evident that h is the only homomorphism of N into K with this property. The proof is therefore complete.

1.5 Symmetric multilinear mappings

Once again M denotes an R-module and all products $M \times M \times \cdots \times M$ and $M \otimes M \otimes \cdots \otimes M$ are understood to have p $(p \geq 1)$ factors.

Let N be an R-module. A multilinear mapping

$$\theta: M \times M \times \cdots \times M \to N$$

is called *symmetric* if

$$\theta(m_1, m_2, \ldots, m_p) = \theta(m_{i_1}, m_{i_2}, \ldots, m_{i_p}), \tag{1.5.1}$$

whenever $m_1, m_2, \ldots, m_p$ belong to M and $(i_1, i_2, \ldots, i_p)$ is a permutation of $(1, 2, \ldots, p)$. Clearly θ is symmetric provided $\theta(m_1, m_2, \ldots, m_p)$ remains unaltered whenever two *adjacent* terms are interchanged.

If θ is a symmetric multilinear mapping and $h: N \to K$ is a homomorphism of R-modules, then $h \circ \theta$ is a symmetric multilinear mapping of $M \times M \times \cdots \times M$ into K. This inevitably prompts the next universal problem.

Problem 3. *To choose* N *and* θ *so that given any symmetric multilinear mapping*