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**PACKING AND COVERING**

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# PACKING AND COVERING

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BY

C. A. ROGERS

D.Sc., F.R.S.

*Astor Professor of Mathematics at  
University College London*



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## PREFACE

Until recently the theory of packing and covering was not sufficiently well developed to justify the publication of a book devoted exclusively to it. After the publication of L. Fejes Tóth's excellent book in 1953, there would be no need for a second work on the subject, but for the fact that he confines his attention mainly to problems in two and three dimensions. Since my own interest has been mainly in the problems which arise in  $n$ -dimensional space, with  $n$  large, this book, which concentrates mainly on problems in  $n$ -dimensional space, is offered as a complement to Fejes Tóth's work. The present seems to me to be an opportune time for the publication of such a book, since I have the impression that most of the simplest general results of the subject have already been discovered, and that further progress may be rather slow, depending on detailed and complicated technical developments. If I am proved wrong in this I shall be happy.

Most mathematical authors are faced with the dilemma that, while the history of a subject usually begins with very special results and only leads gradually up to generalities, an economical and logical exposition of the subject requires that the special results should be obtained as particular cases of the general. This book is the result of a compromise between the two approaches. In the introduction we give a historical outline of the subject, stating results without proof. In the succeeding chapters we give a systematic (but by no means exhaustive) account of the general  $n$ -dimensional results of the subject.

Professor H. Davenport aroused my interest in the subjects of packing and covering when I returned to University College in 1946. I am most conscious of my debt to Professor Davenport in this as in many other ways. I must also express my gratitude to H. Minkowski, G. Voronoi, H. F. Blichfeldt and Professors H. S. M. Coxeter, H. E. Daniels, B. Delaunay, A. M. Macbeath, and Drs L. Few and G. C. Shephard, fragments of whose work will be found reproduced in scarcely disguised form

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PREFACE

in this book. Further, my own work has been much influenced by these mathematicians and by Professors E. Hlawka, K. Mahler, C. L. Siegel and Drs P. Erdős and W. Schmidt.

The idea of writing this book came to me while I was preparing my inaugural lecture on 'Packing and Covering' delivered at University College London in November 1958. The opportunity of making a first draft came when I was invited to give a course of lectures on the subject at the Canadian Mathematical Congress held in the summer of 1959 at Fredericton, New Brunswick. The final manuscript was prepared and typed while I was visiting the University of Toronto and The University of British Columbia, Vancouver for the session 1961–62 with the aid of a Canada Council grant. I am most grateful to these institutions for making this book possible.

C. A. R.

UNIVERSITY COLLEGE LONDON

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