INTRODUCTION

A HISTORICAL OUTLINE OF THE THEORIES OF PACKING AND COVERING

1. Lattice packing of spheres

It is difficult to trace the first significant contribution to the mathematical theory of packing, but perhaps the honour should fall to the work of Gauss† in 1831. Although Lagrange in 1773 had developed the theory of reduction of binary quadratic forms, and found the inequalities satisfied by their coefficients, it was not until Gauss introduced the idea of a lattice in 1831 that Lagrange's results became of significance in the theory of packing.

If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are n linearly independent vectors in n-dimensional Euclidean space, the set $\Lambda = \Lambda(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ of all vectors of the form

$$\mathbf{a} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \ldots + u_n\mathbf{a}_n,$$

where $u_1, u_2, \ldots, u_n$ are arbitrary integers, is called a lattice. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{a}_{n+1}, \ldots$ be an enumeration of the points of such a lattice Λ. A system $\mathscr{K}$ consisting of the translates

$$K + \mathbf{a}_i \quad (i = 1, 2, \ldots)$$

of a given Lebesgue measurable set K, by the vectors of the lattice Λ, is called a lattice packing of K, with lattice Λ, when there is no point of space which is common to two or more of the sets of the system. Such a system $\mathscr{K}$ has a density denoted by $\rho(\mathscr{K})$, which will be defined and discussed in Chapter 1, but which may justifiably be regarded as the proportion of the whole of space covered by the sets of the packing.

When examined from this point of view, Lagrange's results imply that

$$\rho(\mathscr{K}_2) \leqslant \frac{\pi}{\sqrt{12}} = 0\cdot9069 \ldots,$$

† These informal references, as well as more formal ones, will be found in the Bibliography. The dates quoted are, strictly, not those of the results, but those carried by the works in which they were published.

for every lattice packing $\mathscr{K}_2$ of an open circle K_2 in the Euclidean plane. It is easy to verify that

$$\rho(\mathscr{K}_2) = \frac{\pi}{\sqrt{12}},$$

in the case when K_2 is the circle

$$x_1^2 + x_2^2 < 1,$$

and $\mathscr{K}_2$ is the lattice packing of K_2 with lattice Λ generated by the points

$$(2, 0), \quad (1, \sqrt{3}).$$

Thus, if we define a lattice-packing density $\delta_L(K)$, by taking

$$\delta_L(K) = \sup_{\mathscr{K}} \rho(\mathscr{K}),$$

the supremum being over all lattice packings $\mathscr{K}$ of the set K, we have

$$\delta_L(K_2) = \frac{\pi}{\sqrt{12}} = 0{\cdot}9069\ldots, \tag{1}$$

when K_2 is the open unit circle.

Largely because of its connection with the arithmetic minimum of a positive definite quadratic form, much effort has been devoted to the study of $\delta_L(K_n)$, where K_n is the unit sphere in n-dimensional space. In 1831 Seeber, in his book on the reduction of ternary quadratic forms, established a system of inequalities satisfied by reduced forms, and in addition made a conjecture which implies that $\delta_L(K_3) = \pi/\sqrt{18}$. In his review of this book, Gauss deduced Seeber's conjecture from his inequalities, and introduced the geometric interpretation providing the connection with the lattice packings of spheres. The value of $\delta_L(K_n)$ was found by Korkine and Zolotareff in 1872 and 1877 when $n = 4$ or 5, and by Blichfeldt in 1925, 1926 and 1934 when $n = 6$, 7 and 8. In 1944 Mordell showed how the result for $n = 8$ could be very simply deduced from the case when $n = 7$. The exact value of $\delta_L(K_n)$ is unknown for $n \geqslant 9$, but lower bounds for it, which seem reasonably good and which may be exact, have been found by Chaundy in 1946 when $n = 9$ or 10, by Coxeter and Todd in 1951 when $n = 12$, and by Barnes in 1959 when $n = 11$. Some further bounds for moderately large values of n are given by Barnes and Wall (1959).

The results may be summarized in Table 1.

TABLE 1

Dimension	Density of closest lattice packing of a sphere		Reference
2	$\frac{\pi}{2\sqrt{3}}$	0·9069...	Lagrange, 1773; Gauss, 1831
3	$\frac{\pi}{3\sqrt{2}}$	0·7404...	Gauss, 1831
4	$\frac{\pi^2}{16}$	0·6168...	Korkine and Zolotareff, 1872
5	$\frac{\pi^2}{15\sqrt{2}}$	0·4652...	Korkine and Zolotareff, 1877
6	$\frac{\pi^3}{48\sqrt{3}}$	0·3729...	Blichfeldt, 1925
7	$\frac{\pi^3}{105}$	0·2952...	Blichfeldt, 1926
8	$\frac{\pi^4}{384}$	0·2536...	Blichfeldt, 1934
9	$\geqslant \frac{2\pi^4}{945\sqrt{2}}$	$\geqslant$ 0·1457...	Chaundy, 1946
10	$\geqslant \frac{\pi^5}{1920\sqrt{3}}$	$\geqslant$ 0·0920...	Chaundy, 1946
11	$\geqslant \frac{64\pi^5}{19{,}7110\sqrt{3}}$	$\geqslant$ 0·0604...	Barnes, 1959
12	$\geqslant \frac{\pi^6}{19{,}440}$	$\geqslant$ 0·0494...	Coxeter and Todd, 1951

The inequalities for $n = 9$, 10, 11 and 12 are all obtained by calculating the densities of certain carefully chosen lattice packings of spheres, the arrangement being explicitly known in each case. When n is large it seems to be necessary to fall back on indirect arguments. In 1905 Minkowski proved that for all n

$$\delta_L(K_n) \geqslant \zeta(n)/2^{n-1}, \tag{2}$$

where
$$\zeta(n) = \sum_{k=1}^{\infty} k^{-n}.$$

This is a particular case of the more general inequality

$$\delta_L(K) \geqslant \zeta(n)/2^{n-1}, \tag{3}$$

valid for any convex symmetrical body K, which was stated (by implication) by Minkowski in 1893 and proved by Hlawka in 1944. This more general inequality and some of its refinements will be discussed in §2; here we merely draw attention to the improvement

$$\delta_L(K_n) \geqslant \frac{n\zeta(n)}{e(1-e^{-n})\,2^{n-1}} \tag{4}$$

obtained by Rogers in 1947, and to its subsequent refinement by Davenport and Rogers, also in 1947.

The first significant upper bound for $\delta_L(K_n)$ was the bound

$$\frac{n+2}{2}\left(\frac{1}{\sqrt{2}}\right)^n \tag{5}$$

obtained by Blichfeldt in 1914. It was subsequently refined by Blichfeldt himself in 1929, by Rankin in 1947 and by Rogers in 1958. Although we give an account of Rogers's work in Chapter 7 it will be seen (from equation (11) of that chapter) that the improvement is slight when n is large.

In the following diagram we plot the values of the function

$$\frac{\log_e \delta_L(K_n)}{n}$$

for $n = 2, 3, \ldots, 8$, and its lower bounds for $n = 9, 10, 11, 12$, from the table, in comparison with the bounds

$$-\log_e 2 + \frac{1}{n}\log_e \frac{2n\zeta(n)}{e(1-e^{-n})}$$

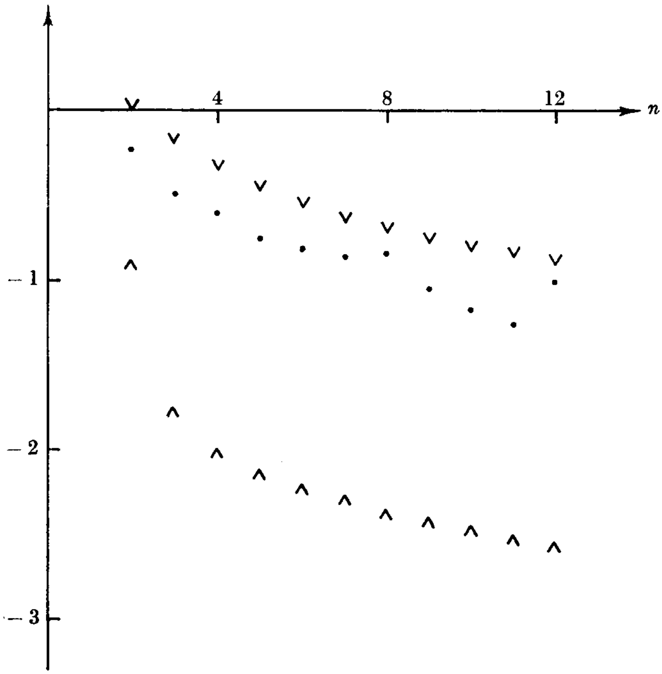
and

$$-\tfrac{1}{2}\log_e 2 + \frac{1}{n}\log_e \frac{n+2}{2}$$

given by (4) and (5). Here we work with (4) and (5) rather than with any of their refinements as they are simple and explicit, while their refinements are complicated and much more difficult to calculate.

For results on 'multiple' lattice packings of spheres and circles see Few (1953), Heppes (1955, 1959) and Blundon (1963).

For results on 'mixed' packings of circles and on 'mixed' lattice packings of 3-dimensional spheres see Fejes Tóth (1953) and Few (1960).



2. Lattice packing of convex sets

The first person to make a systematic study of lattice packings was Minkowski. His first interest was the theory of numbers, and his 'Fundamental Theorem', to which he was led by the study of papers of Dirichlet and Hermite on quadratic forms (see Minkowski, 1893*b*), is a useful theorem in the theory of numbers which he derived from the apparently trivial result that the density of a lattice packing is necessarily less than or equal to 1. Although he made applications of his results to the theory of numbers whenever possible, he evidently became interested in the theory of packing for its own sake. In 1904 he discussed the closest lattice packings of convex sets in three dimensions. He showed generally that in any number of dimensions the problem of determining the closest lattice packing of an asymmetrical convex set could be reduced, at least in theory, to that of determining the closest lattice packing of a symmetrical convex set. If the methods he developed for studying the case $n = 3$ are used in the case $n = 2$ it follows very simply† that the density of the closest lattice packing of a convex plane set K, with the origin $\mathbf{o}$ as centre, is

$$\delta_L(K) = \frac{4\mu(K)}{3h(K)}, \tag{6}$$

where $\mu(K)$ denotes the area of K, and $h(K)$ denotes the area of the largest hexagon, with vertices of the form

$$\mathbf{u},\ \mathbf{v},\ \mathbf{v}-\mathbf{u},\ -\mathbf{u},\ -\mathbf{v},\ -\mathbf{v}+\mathbf{u}$$

lying in this order on the boundary of K. A much more difficult discussion of the case when $n = 3$ led to a theoretical determination of $\delta_L(K)$ for a symmetrical convex 3-dimensional body K, in terms of $\mu(K)$ and the volumes of certain types of convex polyhedra inscribed in K.

In most cases the formula (6) is less convenient than the formula

$$\delta_L(K) = \frac{\mu(K)}{H(K)}, \tag{7}$$

where $H(K)$ denotes the area of the smallest hexagon or quadrilateral (necessarily symmetrical) circumscribed about the symmetrical convex plane set K. This result was discovered by Reinhardt in 1934 and rediscovered by Mahler (1947*b*).

Although Reinhardt's result gives a most satisfying answer to the problem of determining $\delta_L(K)$, when K is a convex symmetrical plane set; the problem of determining the set K of this form, for which $\delta_L(K)$ is least, remains unsolved. Both Reinhardt and Mahler (1947*b*) gave an example of such a set with

$$\delta_L(K) = \frac{8-4\sqrt{2}-\log 2}{2\sqrt{2}-1} = 0{\cdot}9024\ldots.$$

† But although the result must have been familiar to Minkowski, I have not found it in his published work.

Mahler (1946*b*) proved that all such sets satisfy

$$\delta_L(K) \geqslant \sqrt{(\tfrac{3}{4})} = 0{\cdot}8660\ldots.$$

More recently Ennola (1961) has obtained the stronger inequality

$$\delta_L(K) \geqslant \tfrac{1}{4}\{3\sqrt{2}+\sqrt{3}-\sqrt{6}\} = 0{\cdot}8813\ldots \tag{8}$$

and announced the lower bound 0·8925....

Despite considerable theoretical advances in the Geometry of Numbers since Minkowski's time (see Cassels, 1959), the problem of determining the value of $\delta_L(K)$ for a given convex 3-dimensional body K remains a formidable task. Minkowski himself in 1904 showed that

$$\delta_L(K) = \tfrac{18}{19} = 0{\cdot}9473\ldots,$$

when K is the octahedron defined by

$$|x_1|+|x_2|+|x_3| < 1.$$

Whitworth (1948, 1951) used Minkowski's methods to obtain the value when K is a cube with two opposite corners truncated by plane faces and when K is a 'double cone'. Chalk in 1950 found the density when K is a 'slice' cut from a sphere. Mahler (1946*a*; see also Hlawka, 1948) obtained the density, when K is a circular cylinder, by using the theory of (general) packings of circles. This result was extended to cylinders on an arbitrary convex plane base by Chalk and Rogers (1948) and Yeh (1948), and to cylinders on a base consisting of a 3-dimensional sphere by Woods in 1958.

Apart from the results for spheres discussed in § 1, certain results for cylinders, and certain examples of space-filling sets, the exact value of $\delta_L(K)$ remains unknown for all convex sets K in 4 or more dimensions. So, when $n \geqslant 4$, the main interest lies in the determination of lower bounds for $\delta_L(K)$ for various classes of sets K and estimates for $\delta_L(K)$ for certain special sets K, the spheres and the simplices being perhaps the most interesting.

Minkowski (1893*a*) announced a result in the Geometry of Numbers concerning star bodies. Application of this result to a convex symmetrical n-dimensional set K immediately yields the inequality

$$\delta_L(K) \geqslant \zeta(n)/2^{n-1} \quad \left(\zeta(n) = \sum_{k=1}^{\infty} k^{-n}\right). \tag{9}$$

Minkowski only published a proof of his result in 1905 in the special case when K is a sphere or ellipsoid. Blichfeldt† in 1919 stated that he had obtained a stronger inequality in the case when K is convex and symmetrical. In 1944 Hlawka published the first complete proof of Minkowski's theorem, and Mahler published (a few months later) a proof of a result which was only slightly weaker. In 1945 Siegel published a proof making use of the rather deep considerations that Minkowski had used in discussing the case of an ellipsoid; it seems likely that Siegel's proof is similar to Minkowski's unpublished work; it has the advantage that it works with a natural measure in the space of lattices. Since 1945 many proofs and refinements‡ of the Minkowski–Hlawka theorem have been published. The first refinement was obtained by Mahler; most of the more recent ones are due to Rogers and Schmidt. Schmidt (1958) was the first to obtain an inequality of the form

$$\delta_L(K) \geqslant cn/2^n, \tag{10}$$

valid for all n and a suitable constant c; in his latest paper on the subject he shows that this inequality holds for convex symmetrical K when n is sufficiently large provided

$$c < \log 2. \tag{11}$$

The proofs of the more refined inequalities are exceedingly complicated. In Chapter 4 we give only the simplest approach to the problem, proving that

$$\delta_L(K) \geqslant 1/2^{n-1} \tag{12}$$

for each convex symmetrical n-dimensional set K. Since

$$\zeta(n) = 1 + O((\tfrac{1}{2})^n)$$

† See also Bernstein (1918).

‡ Bateman (1962), Cassels (1953), Davenport and Rogers (1947), Lekkerkerker (1956), Macbeath and Rogers (1955, 1958 *a*, *b*), Mahler (1946), Malyšev (1952), Rogers (1947, 1951 *a*, 1954, 1955 *a*, *b*, 1956 *a*, *b*, 1957 *b*, 1958 *b*), Sanov (1952), Santaló (1950), Schmidt (1956 *a*, *b*, 1957, 1958 *a*, *b*, 1959, 1963), Weil (1946).

this inequality is only fractionally weaker than (9) when n is large.

There remains a wide gap between the results of the Minkowski–Hlawka type, showing that for each convex symmetrical set K the lattice-packing density is at least $cn/2^n$, and the results of Blichfeldt type, showing that the lattice-packing density (and indeed the packing density) is less than $cn/(\sqrt{2})^n$ for an n-dimensional sphere.

So far we have been mainly concerned with the lattice packing of symmetrical convex sets. When we turn to convex sets that are not necessarily symmetrical, we find that despite Minkowski's reduction of the general case to the symmetrical case very little is known. If K is any set, its difference set $\mathsf{D}K$ is defined to be the set of all points of the form

$$\mathbf{x}-\mathbf{y}$$

with $\mathbf{x}$ and $\mathbf{y}$ in K. It is easy to verify that $\mathsf{D}K$ is automatically symmetrical in $\mathbf{o}$ and that $\mathsf{D}K$ is convex if K is convex. Provided K is convex, Minkowski's argument (see Chapter 6, Theorem 6.7) shows that

$$\frac{\delta_L(K)}{\mu(K)} = \frac{\delta_L(\mathsf{D}K)}{\mu(\mathsf{D}K)}2^n. \tag{13}$$

In 1904 Minkowski claimed to have determined $\delta_L(K)$, when K is a tetrahedron, in this way, but he was plainly wrong in asserting that the difference body of a tetrahedron was an octahedron.†

When $n = 2$ the relationship between K and $\mathsf{D}K$ is not difficult to investigate, and Fáry in 1950 was able to show that, for all open convex plane sets K,

$$\delta_L(K) \geqslant \tfrac{2}{3},$$

with equality only when K is a triangle. For $n \geqslant 3$ the key to progress is to find the upper bound of the ratio

$$\frac{\mu(\mathsf{D}K)}{\mu(K)}.$$

† For Pepper's discussion of this point see Hancock (1939).

In 1925 Rademacher showed that the bound was 6 when n is 2. In 1928 Estermann and Süss independently obtained the result 20 when $n = 3$. The natural conjecture is that the exact upper bound is the binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$

in general, and this was proved by Rogers and Shephard in 1957 and in a more geometrical way in 1958 (see Chapter 2, §2). Combining this with (12) and (13) we immediately have

$$\delta_L(K) \geqslant \frac{2(n!)^2}{(2n)!} \tag{14}$$

for any convex set K; we obtain this inequality in a slightly different way in Theorem 4.4 of Chapter 4.

Since

$$\frac{2(n!)^2}{(2n)!} \sim \frac{2\sqrt{(\pi n)}}{4^n} \quad (n \to \infty),$$

the inequality (14) is very weak when n is large. Rogers and Shephard (1957) remark that the right-hand side of any equality of the form (14) is necessarily very small, since in the case of a simplex

$$\delta_L(K) \leqslant \frac{2^n(n!)^2}{(2n)!} \sim \frac{\sqrt{(\pi n)}}{2^n} \quad (n \to \infty). \tag{15}$$

An account of this is given in Chapter 6.

3. Packing of convex sets

As a natural extension of the ideas of §1 a system $\mathscr{K}$ consisting of the translates

$$K + \mathbf{a}_i \quad (i = 1, 2, \ldots)$$

of a given Lebesgue measurable set K, by the vectors of a sequence $\mathbf{a}_1$, $\mathbf{a}_2$, ..., which may be finite or infinite, is called a packing of K when there is no point of space which is common to two or more of the sets of the system. If the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ ... are distributed in a sufficiently regular way throughout the whole of space (in particular, for example, if the set of points $\{\mathbf{a}_i\}$ is periodic with some period in each coordinate) we can assign a density $\rho(\mathscr{K})$ to the system. For a detailed discussion see