1

THEORY OF SETS

1.1 Sets

We do not want to become involved in the logical foundations of mathematics. In order to avoid these we will adopt a rather naïve attitude to set theory. This will not lead us into difficulties because in any given situation we will be considering sets which are all contained in (are subsets of) a fixed set or space or suitable collections of such sets. The logical difficulties which can arise in set theory only appear when one considers sets which are 'too big'—like the set of all sets, for instance. We assume the basic algebraic properties of the positive integers, the real numbers, and Euclidean spaces and make no attempt to obtain these from more primitive set theoretic notions. However, we will give an outline development (in Chapter 2) of the topological properties of these sets.

In a space X a set E is well defined if there is a rule which determines, for each *element* (or point) x in X, whether or not it is in E. We write $x \in E$ (read 'x belongs to E') whenever x is an element of E, and the negation of this statement is written $x \notin E$. Given two sets E, F we say that E is contained in F, or E is a subset of F, or F contains E and write $E \subset F$ if every element x in E also belongs to F. If $E \subset F$ and there is at least one element in F but not in E, we say that E is a proper subset of F.

Two sets E, F are *equal* if and only if they contain the same elements; i.e. if and only if $E \subset F$ *and* $F \subset E$. In this case we write $E = F$. This means that if we want to prove that $E = F$ we must prove both $x \in E \Rightarrow x \in F$ and $x \in F \Rightarrow x \in E$ (the symbol $\Rightarrow$ should be read 'implies').

Since a set is determined by its elements, one of the commonest methods of describing a set is by means of a defining sentence: thus E is the set of all elements (of X) which have the property P (usually delineated). The notation of 'braces' is often used in this situation

$$E = \{x: x \text{ has property } P\}$$

but when we use this notation we will always assume that only elements x in some fixed set X are being considered—as otherwise logical paradoxes can arise. When a set has only a finite number of elements we can write them down between braces $E = \{x, y, z, a, b\}$. In particular $\{x\}$ stands for the set containing the single element x. One must always distinguish between the element x and the set $\{x\}$, for example, the empty set $\varnothing$ defined below is not the same as the class $\{\varnothing\}$ containing the empty set.

Empty set (*or null set*)

The set which contains no elements is called the empty set and will be denoted by $\varnothing$. Clearly

$$\varnothing = \{x : x \neq x\}, \quad \text{and} \quad \varnothing \subset E \text{ for all sets } E.$$

In fact since $\varnothing$ contains no element, any statement made about the elements of $\varnothing$ is true (as well as its negative).

There are some sets which will be considered very frequently, and we consistently use the following notation:

$\mathbf{Z}$, for the set of positive integers,
$\mathbf{Q}$, for the set of rationals,
$\mathbf{R} = \mathbf{R}^1$, for the set of all real numbers,
$\mathbf{C}$, for the set of complex numbers,
$\mathbf{R}^n$, for Euclidean n-dimensional space, i.e. the set of ordered n-tuples $(x_1, x_2, \ldots, x_n)$ where all the x_i are in $\mathbf{R}$.

We assume that the reader is familiar with the algebraic and order properties of these sets. In particular we will use the fact that $\mathbf{Z}$ is *well ordered*, that is, that every non-empty set of positive integers has a least member: this is equivalent to the principle of mathematical induction.

We frequently have to consider sets of sets, and occasionally sets of sets of sets. It is convenient to talk of *classes* of sets and *collections* of classes to distinguish these types of set, and we will use italic capitals $A, B, \ldots$ for sets, script capitals $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ for classes and Greek capitals $\Delta, \Gamma, \ldots$ for collections. Thus $C \in \mathscr{C}$ is read 'the set C belongs to the class $\mathscr{C}$'; and $\mathscr{A} \subset \mathscr{B}$ means that every set in the class $\mathscr{A}$ is also in the class $\mathscr{B}$.

Cartesian product

Given two sets E, F we define the Cartesian (or direct) product $E \times F$ to be the set of all *ordered* pairs (x, y) whose first element $x \in E$ and whose second element $y \in F$. This clearly extends immediately to the product $E_1 \times E_2 \times \ldots \times E_n$ of any finite number of sets. In particular it is immediate that $\mathbf{R}^n$, Euclidean n-space, is the Cartesian product of n copies of $\mathbf{R}$. For an infinite indexed class $\{E_i, i \in I\}$ of sets, the product $\prod_{i \in I} E_i$ is the set of elements of the form $\{a_i, i \in I\}$ with $a_i \in E_i$ for each $i \in I$.

Exercises 1.1

1. Describe in words the following sets:

(i) $\{t \in \mathbf{R}: 0 \leqslant t \leqslant 1\}$;
(ii) $\{(x, y) \in \mathbf{R}^2: x^2 + y^2 \leqslant 1\}$;
(iii) $\{k \in \mathbf{Z}: k = n^2 \text{ for some } n \in \mathbf{Z}\}$;
(iv) $\{k \in \mathbf{Z}: n | k \Rightarrow n = 1 \text{ or } k\}$;
(v) $\{\mathscr{A}: E \in \mathscr{A}\}$;
(vi) $\{B: B \subset E\}$.

2. Show that the relation $\subset$ is reflexive and transitive, but not in general symmetric.

3. The sets $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are different but there is a natural correspondence between them.

4. Suppose x is an element of X and $A = \{x\}$. Which of the following statements are correct: $x \in A$, $x \in X$, $x \subset A$, $x \subset X$, $A \in X$, $A \subset X$, $A \subset x$?

5. Suppose $P(\alpha)$ and $Q(\alpha)$ are two propositions about the element such that $P(\alpha) \Rightarrow Q(\alpha)$. Show that $\{\alpha: P(\alpha)\} \subset \{\alpha: Q(\alpha)\}$.

1.2 Mappings

Suppose A and B are any two sets: a *function* from A to B is a rule which, for each element in A, determines a unique element in B. We talk of the function f and use the notation $f: A \to B$ to denote a function f defined on A and taking values in B. For any $x \in A$, $f(x)$ means the value of the function f at the point x and is therefore an element of the set B: we therefore avoid the terminology (common in older text books) 'the function $f(x)$'. The words *mapping* and *transformation* are often used as a synonym for function.

For a given function $f: A \to B$, we call A the *domain* of f and the subset of B consisting of the set of values $f(x)$ for x in A is called the *range* of f and may be denoted $f(A)$. When $f(A) = B$ we say that f is a function from A *onto* B. Given a function $f: A \to B$, by definition $f(x)$ is a uniquely determined element of B for each $x \in A$; if in addition for each y in $f(A)$ there is a *unique* $x \in A$ (we know there is at least one) with $y = f(x)$ we say that the function f is $(1, 1)$. Another shorter way of saying this is that $f: A \to B$ is $(1, 1)$ if and only if for $x_1, x_2 \in A$,

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

Given $f: A \to B$ there is an associated $f: \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}$ is the class of all subsets of A and $\mathscr{B}$ is the class of all subsets of B, defined by

$$f(E) = \{y \in B: \exists x \in E \quad \text{with} \quad y = f(x)\}$$

for each $E \subset A$. (the symbol $\exists$ should be read, 'there exists': i.e. the set described by $\{x \in E: y = f(x)\}$ is not empty). There is also a function $f^{-1}: \mathscr{B} \to \mathscr{A}$ defined by

$$f^{-1}(F) = \{x \in A: f(x) \in F\},$$

for each $F \subset B$. The set $f^{-1}(F)$ is called the *inverse image* of F under f. Note that if $y \in B - f(A)$, then the inverse image $f^{-1}(\{y\})$ of the one point set $\{y\}$ is the empty set. If $f: A \to B$ is $(1, 1)$ and $y \in f(A)$, then it is clear that $f^{-1}(\{y\})$ is a one point subset of A, so that in this case (only) we can think of f^{-1} as a function from $f(A)$ to A. In particular, if $f: A \to B$ is $(1, 1)$ and onto there is a function $f^{-1}: B \to A$ called the *inverse function* of f such that $f^{-1}(y) = x$ if and only if $y = f(x)$.

Now suppose $f: A_1 \to B$, $g: A_2 \to B$ are functions such that $A_1 \supset A_2$ and $f(x) = g(x)$ for all x in A_2: under these conditions we say that f is an *extension* of g (from A_2 to A_1) and g is the *restriction* of f (to A_2). For example, if

$$g(x) = \cos x \quad (x \in \mathbf{R});$$

$$f(x+iy) = \cos x \cosh y + i \sin x \sinh y \quad (x + iy \in \mathbf{C});$$

then $f: \mathbf{C} \to \mathbf{C}$ is an extension of $g: \mathbf{R} \to \mathbf{C}$ from $\mathbf{R}$ to $\mathbf{C}$, and the usual convention of designating both f and g by 'cos' obscures the differences in their domains.

If we have two functions $f: A \to B$, $g: B \to C$ the result of applying the rule for g to the element $f(x)$ defines an element in C for all $x \in A$. Thus we have defined a function $h: A \to C$ which is called the *composition* of f and g and denoted $g \circ f$ or $g(f)$. Thus, for $x \in A$

$$h(x) = (g \circ f)\, x = g(f(x)) \in C.$$

Note that, if $f: A \to B$ is $(1, 1)$ and onto we could define the inverse function $f^{-1}: B \to A$ as the unique function from B to A such that

$$(f \circ f^{-1})\,(y) = y \quad \text{for all} \quad y \in B,$$

$$(f^{-1} \circ f)\,(x) = x \quad \text{for all} \quad x \in A.$$

Sequence

Given any set X a *finite sequence* of n points of X is a function from $\{1, 2, \ldots, n\}$ to X. This is usually denoted by $x_1, x_2, \ldots, x_n$ where $x_i \in X$ is the value of the function at the integer i. Similarly, an *infinite* *sequence* in X is a function from $\mathbf{Z}$ to X (where $\mathbf{Z}$ is the set of positive integers). This is denoted $x_1, x_2, \ldots$, or $\{x_i\}$ $(i = 1, 2, \ldots)$, or just $\{x_i\}$ where x_i is the value of the function at i, and is called the ith element of the sequence. Given a sequence $\{n_i\}$ of positive integers (that is, a function $f: \mathbf{Z} \to \mathbf{Z}$ where $f(i) = n_i$) such that $n_i > n_j$ for $i > j$, and a sequence $\{x_i\}$ of elements of X (a function $g: \mathbf{Z} \to X$) it is clear that the composite function $g \circ f: \mathbf{Z} \to X$ is again a sequence. Such a sequence is called a *subsequence* of $\{x_i\}$ and is denoted $\{x_{n_i}\}$ $(i = 1, 2, \ldots)$. Thus $\{x_{n_i}\}$ is a subsequence of $\{x_i\}$ if $n_i \in \mathbf{Z}$ for all $i \in \mathbf{Z}$, and $i > j \Rightarrow n_i > n_j$.

We can think of a sequence as a point in the product space $\prod_{i=1}^{\infty} X_i$ where $X_i = X$ for all i. More generally a point in the product space $\prod_{i \in I} X_i$ with $X_i = X$ for $i \in I$ can be identified as a function $f: I \to X$.

Exercises 1.2

1. Suppose $f: \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = \sin x$. Describe each of the following sets:

$$f^{-1}\{0\}, \quad f^{-1}\{1\}, \quad f^{-1}\{2\}, \quad f^{-1}\{y: 0 \leqslant y \leqslant \tfrac{1}{2}\}.$$

2. Suppose $f: A \to B$ is any function. Prove

(i) $E \subset f^{-1}(f(E))$, for each $E \subset A$;
(ii) $F \supset f(f^{-1}(F))$, for each $F \subset B$;

and give examples in which there is not equality in (i), (ii).

3. Suppose $f: A \to B$, $g: B \to C$ are functions and $h = g \circ f$: show that $h^{-1}(E) = f^{-1}[g^{-1}(E)]$ for each $E \subset C$.

4. If $A \subset B \subset C$, $f: A \to X$, $g: B \to X$, $h: C \to X$ are such that h is an extension of g and g is an extension of f, prove that f is the restriction of h to A.

5. Show that the restriction of a $(1, 1)$ mapping is $(1, 1)$.

6. Suppose $m, n \in \mathbf{Z}$, A is a set with m distinct elements and B is a set with n distinct elements. How many distinct functions are there from A to B?

1.3 Cardinal numbers

If there is a mapping $f: A \to B$ which is $(1, 1)$ and onto, then it is reasonable to say that there are the same number of elements in A as there are in B. In fact, for finite sets, the elementary process of counting sets up such a mapping from the set being counted to the integers $\{1, 2, \ldots, n\}$, and from experience we know that if the same finite set of objects is counted in different ways we always end up with the same integer n. (This fact can also be deduced from primitive axioms about the integers.) We say that the set A is equivalent to the set B, and write $A \sim B$ if there is a mapping $f: A \to B$ which is (1, 1) and onto. It is clear that $\sim$ is an equivalence relation between sets in the sense that it is reflexive, symmetric and transitive, and we can therefore form equivalence classes of sets with respect to this relation. Such an equivalence class of sets is called a *cardinal number*, but by noting that the equivalence class is determined by any one of its members, we see that the easiest way to specify a cardinal number is to specify a representative set. Thus any set which can be mapped (1, 1) onto the representative set will have the same cardinal. As is usual we shall use the following notation:

the cardinal of the empty set $\varnothing$ is 0;
the cardinal of the set of integers $\{1, 2, \ldots n\}$ is n;
the cardinal of the set $\mathbf{Z}$ of positive integers is $\aleph_0$;
the cardinal of the set $\mathbf{R}$ of real numbers is c.

Since $\mathbf{Z}$ is ordered we can clearly order the cardinals of *finite* sets by saying that A has a smaller cardinal than B if A is equivalent to a proper subset of B. This definition does not work for infinite sets as the mappings

$$n \to 2^n \quad \text{or} \quad n \to n^2$$

map $\mathbf{Z}$ onto a proper subset of $\mathbf{Z}$ and are (1, 1). Instead we say that the cardinal of a set A is less than the cardinal of the set B if there is a subset $B_1 \subset B$ such that $A \sim B_1$ but no subset $A_1 \subset A$ such that $A_1 \sim B$.

From this definition of ordering we consider the following statements, where m, n, p denote cardinals

(i) $m < n, n < p \Rightarrow m < p$;

(ii) at *most* one of the relations $m < n$, $m = n$, $n < m$ holds so that $m \leqslant n, n \leqslant m \Rightarrow m = n$.

(iii) at *least* one of the relations $m < n$, $m = n$, $n < m$ holds.

Now (i) follows easily from the definition, for let M, N, P be sets with cardinals m, n, p and suppose $N_1 \subset N$, $P_1 \subset P$ with $M \sim N_1$, $N \sim P_1$. The mapping $f: N \to P_1$ when restricted to N_1 gives an equivalence $N_1 \sim P_2 \subset P_1$ so that $M \sim P_2 \subset P$. Further if $P \sim M_1 \subset M$ the mapping $g: M \to N_1$ when restricted to M_1 shows $P \sim M_1 \sim N_2 \subset N$ which contradicts $n < p$. (ii) can also be deduced from the definition (see exercise 1.3 (5)), though this requires quite a complicated argument: (ii) is known as the Schröder–Bernstein theorem. However, the truth of (iii)—that all cardinals are comparable—cannot be proved without the use of an additional axiom (known as the axiom of choice) which we will discuss briefly in §1.6. If we assume the axiom of choice or something equivalent, then (iii) is also true.

A set of cardinal $\aleph_0$ is said to be *enumerable*. Thus such a set $A \sim \mathbf{Z}$ so that the elements of A can be 'enumerated' as a sequence $a_1, a_2, \ldots$ in which each element of A occurs once and only once. A set which has a cardinal $m \leqslant \aleph_0$ is said to be *countable*. Thus E is countable if there is a subset $A \subset \mathbf{Z}$ such that $E \sim A$, and a set is countable if it is either finite or enumerable.

Given any infinite set B we can choose, by induction, a sequence $\{b_i\}$ of distinct elements in B and if B_1 is the set of elements in $\{b_i\}$ the cardinal of B_1 is $\aleph_0$. Hence if m is an infinite cardinal we always have $m \geqslant \aleph_0$. By using the equivalence

$$b_i \leftrightarrow b_{2i}$$

between B_1 and the proper subset $B_2 \subset B_1$ where B_2 contains the even elements of $\{b_i\}$ and the identity mapping

$$b \leftrightarrow b \quad \text{for} \quad b \in B - B_1$$

we have an equivalence between $B = B_1 \cup (B - B_1)$ and $B_2 \cup (B - B_1)$, a proper subset of B. This shows that any infinite set B contains a proper subset of the same cardinal.

In order to see that some infinite sets have cardinal $> \aleph_0$ it is sufficient to recall that the set $\{x \in \mathbf{R}: 0 < x < 1\}$ cannot be arranged as a sequence.† Now $\pi^{-1} \tan^{-1} x + \frac{1}{2} = f(x)$, $x \in \mathbf{R}$ defines a mapping f: $\mathbf{R} \to (0, 1)$ which is $(1, 1)$ and onto so that $\mathbf{R}$ has the same cardinal as the interval $(0, 1)$ and we have $\mathbf{c} > \aleph_0$. It is worth remarking that a famous unsolved problem of mathematics concerns the existence or otherwise of cardinals m such that $\mathbf{c} > m > \aleph_0$. The axiom that no such exist, that is that $m > \aleph_0 \Rightarrow m \geqslant \mathbf{c}$ is known as the *continuum hypothesis*.

The fact that there are infinitely many different infinite cardinals follows from the next theorem, which compares the cardinal of a set E with the cardinal of the class of subsets of E.

***Theorem* 1.1.** *For any set E, the class $\mathscr{C} = \mathscr{C}(E)$ of all subsets of E has a cardinal greater than that of E.*

Proof. For sets E of finite cardinal n, one can prove directly that the cardinal of $\mathscr{C}(E)$ is 2^n, and an induction argument easily yields $n < 2^n$ for $n \in \mathbf{Z}$. However, the case of finite sets E is included in the general proof, so there is nothing gained by this special argument.

† See, for example, J. C. Burkill, *A First Course in Mathematical Analysis* (Cambridge, 1962).

Suppose $\mathscr{Q}$ is the class of one points sets $\{x\}$ with $x \in E$. Then $\mathscr{Q} \subset \mathscr{C}$ and $E \sim \mathscr{Q}$ because of the mapping $x \leftrightarrow \{x\}$. Therefore it is sufficient to prove by (ii) above, that $\mathscr{C}$ is equivalent to no subset $E_1 \subset E$. Suppose then that $\phi: \mathscr{C} \to E_1$ is (1, 1) and onto and let $\chi: E_1 \to \mathscr{C}$ denote the inverse function. Let A be the subset of E_1 defined by

$$A = \{x \in E_1, \quad x \notin \chi(x)\}.$$

Then $A \in \mathscr{C}$ so that $\phi(A) = x_0 \in E_1$. Now if $x_0 \in A$, $\chi(x_0) = A$ does not contain x_0 which is impossible, while if $x_0 \notin A$, then x_0 is not in $\chi(x_0)$ so that $x_0 \in A$. In either case we have a contradiction. ∎

It is possible to build up systematically an arithmetic of cardinals. This will only be needed for finite cardinals and $\aleph_0$ in this book, so we restrict the results to these cases and discuss them in the next section.

Exercises 1.3

1. Show that $(0, 1] \sim (0, 1)$ by considering f defined by

$$\begin{aligned} f(x) &= \tfrac{3}{2} - x, \quad \text{for} \quad \tfrac{1}{2} < x \leqslant 1; \\ &= \tfrac{3}{4} - x, \quad \text{for} \quad \tfrac{1}{4} < x \leqslant \tfrac{1}{2}; \\ &= \tfrac{3}{8} - x, \quad \text{for} \quad \tfrac{1}{8} < x \leqslant \tfrac{1}{4}; \\ &= \frac{3}{2^n} - x, \quad \text{for} \quad \frac{1}{2^n} < x \leqslant \frac{1}{2^{n-1}} \quad (n = 1, 2, \ldots). \end{aligned}$$

Deduce that all intervals (a, b), $(a, b]$, $[a, b]$ or $[a, b)$ with $a < b$ have the same cardinal $\mathbf{c}$.

2. Every function $f: [a, b] \to \mathbf{R}$ which is monotonic, i.e.

$$a \leqslant x_1 < x_2 \leqslant b \Rightarrow f(x_1) \leqslant f(x_2),$$

is discontinuous at the points of a countable subset of $[a, b]$.

Hint. Consider the sets of points x where the size of the discontinuity $d(x) = f(x+0) - f(x-0)$ satisfies $1/(n+1) < d(x) \leqslant 1/n$ and prove this is finite for all n in $\mathbf{Z}$.

3. Show that $\mathbf{R}^2 \sim \mathbf{R}$.

Hint.

$$(\cdot a_1 a_2 a_3 \ldots, \cdot b_1 b_2 b_3 \ldots) \leftrightarrow \cdot a_1 b_1 a_2 b_2 \ldots$$

defines a (1, 1) mapping between pairs of decimal expansions and single expansions of numbers in (0, 1). Modify this mapping to eliminate the difficulty caused by the fact that decimal expansions are not quite unique.

4. Prove that a finite set E of cardinal m has 2^m distinct subsets.

5. Suppose $A_1 \subset A$, $B_1 \subset B$, $A_1 \sim B$ and $A \sim B_1$. Construct a mapping to show that $A \sim B$.

Hint. Suppose $f: A \to B_1$, $g: B \to A_1$ are (1, 1) and onto. Say x (in either A or B) is an ancestor of y if and only if y can be obtained from x by successive applications of f and g. Decompose A into 3 sets A_o, A_e, A_i according as to whether the element x has an odd, even or infinite number of ancestors and decompose B similarly. Consider the mapping which agrees with f on A_e and A_i, and with g^{-1} on A_o.

1.4 Operations on subsets

For two sets A, B we define the *union* of A and B (denoted $A \cup B$) to be the set of elements in either A or B or both. The *intersection* of A and B (denoted $A \cap B$) is the set of elements in both A and B.

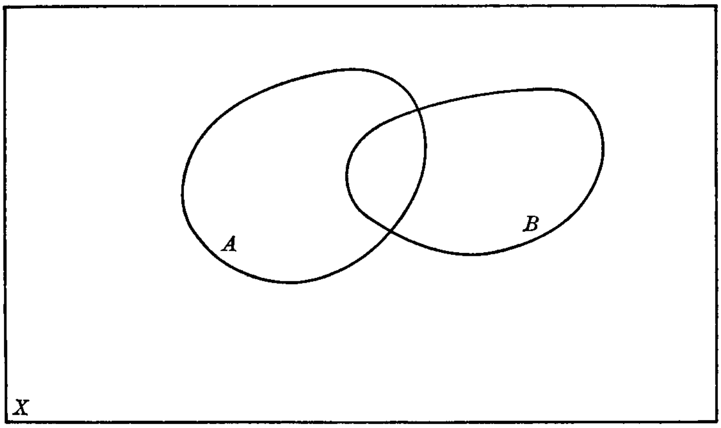

Fig. 1

If $A \subset X$, the *complement* of A with respect to X (denoted $X - A$) is the set of those elements in X which are not in A. We also use $(A - B)$ to denote the set of elements in A which are not in B for arbitrary sets A, B. For any two sets A, B the *symmetric difference* (denoted $A \triangle B$) is $(A - B) \cup (B - A)$, that is the set of elements which are in one of A, B but not in both. Note that $A \triangle B = B \triangle A$.

These finite operations on sets are best illustrated by means of a Venn diagram. In this some figure (like a rectangle) denotes the whole space X and suitable geometrical figures inside denote the subsets A, B, etc. It is well known that drawing does not prove a theorem, but the reader is advised to illustrate the results of the next paragraph by means of suitable Venn diagrams (see Figure 1).

The operations $\cup, \cap, \triangle$ satisfy algebraic laws, some of which are listed below. We assume the reader is familiar with these, so proofs are omitted.

(i) $A \cup B = B \cup A, \quad A \cap B = B \cap A;$

(ii) $(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C);$

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$

(iv) $A \cup \varnothing = A, \quad A \cap \varnothing = \varnothing;$

(v) if $A \subset X$, then $A \cup X = X, \quad A \cap X = A;$

(vi) if $A \subset X$, $B \subset X$, then $X - (A \cup B) = (X - A) \cap (X - B),$
$X - (A \cap B) = (X - A) \cup (X - B);$

(vii) $A \cup B = (A \triangle B) \triangle (A \cap B), \quad A - B = A \triangle (A \cap B).$

A similarity between the laws satisfied by $\cap$, $\cup$ and the usual algebraic laws for multiplication and addition can be observed (in fact the older notation for these operations is product and sum) but the differences should also be noted: in particular the distributive laws, (iii) above, are different in the algebra of sets. (vi) above will be generalized and proved as a lemma—it is known as de Morgan's law.

Given a class $\mathscr{C}$ of subsets A, the *union* $\bigcup\{A; A \in \mathscr{C}\}$ is the set of elements which are in at least one set A belonging to $\mathscr{C}$ and the *intersection* $\bigcap\{A; A \in \mathscr{C}\}$ is the set of elements which are in every set A of $\mathscr{C}$. If the class $\mathscr{C}$ is indexed so that $\mathscr{C}$ consists precisely of the sets A_α, $(\alpha \in I)$, then we use the notations $\bigcup_{\alpha \in I} A_\alpha$, $\bigcap_{\alpha \in I} A_\alpha$ for the union and intersection of the class. In particular when $\mathscr{C}$ is finite or enumerable it is usual to assume that it is indexed by $\{1, 2, \ldots, n\}$ or $\mathbf{Z}$ respectively and the notation is

$$\bigcup_{i=1}^{n} A_i, \quad \bigcap_{i=1}^{n} A_i, \quad \bigcup_{i=1}^{\infty} A_i, \quad \bigcap_{i=1}^{\infty} A_i.$$

When the class $\mathscr{C}$ is empty, that is $I = \varnothing$, we adopt the conventions

$$\bigcup_{\alpha \in I} E_\alpha = \varnothing, \quad \bigcap_{\alpha \in I} E_\alpha = X, \quad \text{the whole space.}$$

This ensures that certain identities are valid without restriction on I.

Lemma. *Suppose $E_\alpha, \alpha \in I$ is a class of subsets of X, and E_1 is one set of the class, then*

(i) $\bigcap_{\alpha \in I} E_\alpha \subset E_1 \subset \bigcup_{\alpha \in I} E_\alpha;$

(ii) $X - \bigcup_{\alpha \in I} E_\alpha = \bigcap_{\alpha \in I} (X - E_\alpha);$

(iii) $X - \bigcap_{\alpha \in I} E_\alpha = \bigcup_{\alpha \in I} (X - E_\alpha).$