Topological Riesz Spaces and Measure Theory

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Acknowledgements

The gestation of this book has extended over three appointments: as a lecturer at United College, The Chinese University of Hong Kong; as a Central Electricity Generating Board Junior Research Fellow at Churchill College, Cambridge; and as a lecturer at the University of Essex. I am grateful for the material support of all these institutions, and for the moral support of my colleagues there.

Almost every mathematician I have spoken to in the last six years has influenced the book in some way; but I should like to give special thanks to Professor W. A. J. Luxemburg, who introduced me to its subject matter; to Dr N. N. Chan, who suggested that it should be written; and to Dr F. Smithies, who gently moulded it into publishable form. Finally I must mention Miss M. Mitchell, who typed the bulk of the MS, and the editors of the Cambridge University Press, for their skill and patience in transforming an idiosyncratic manuscript into print.

Preface

This book is addressed to functional analysts who would like to understand better the application of their subject to the older discipline of measure theory. The relationship of the two subjects has not always been easy. Measure theory has been the source of many examples for functional analysis; and these examples have been leading cases for some of the most important developments of the general theory. Such a stimulation is, of course, entirely welcome. But there have in addition been several cases in which special results in measure theory have been applied to prove general theorems in analysis. The ordinary functional analyst feels inadequately prepared for these applications, and is exasperated by the intrusion of a large body of knowledge in an unfamiliar style into his own concerns.

My aim therefore is to identify those concepts in measure theory which have most affected functional analysis, and to integrate them into the latter subject in a way consistent with its own structure and habits of thought. The most powerful idea is undoubtedly that of Riesz space, or vector lattice. The principal Banach spaces which measure theory has contributed to functional analysis all have natural partial orderings which render them Riesz spaces. Many of their special properties can be related to the ways in which their orderings, their linear structures and their topologies are connected. For a clear understanding of the difference, for instance, between an L^1 space and an L^∞ space, there is no substitute for an abstract analysis of their properties as ordered linear topological spaces.

The other point at which measure theory has had an impact on functional analysis is in the representation of linear functionals as integrals, and the consequent deduction of surprising properties. I think that the analyst's instinctive rejection of such methods is a perfectly sound reaction. The difficulty is to provide techniques powerful enough to act as substitutes. Here again I believe that a thorough understanding of some quite simple topological Riesz spaces can see us through most of the difficulties.

Now as soon as we begin to look at measure theory with a sceptical and abstract eye, a number of peculiarities strike us. The first is the relative unimportance of measure spaces themselves. They are used to construct spaces of equivalence classes of functions which then take on a life of their own; and quite different measure spaces can give rise to isomorphic function spaces. So it is natural to ask just what it is about a measure space that determines the associated function spaces. The answer is readily to hand: it is a construction called the measure algebra. Two measure spaces with isomorphic measure algebras will have isomorphic L^1 spaces, isomorphic L^∞ spaces, and so on; and to a large extent the converse is true.

I think that the most convincing way of demonstrating these facts is to exhibit methods of constructing the function spaces (or, rather, isomorphic copies of them) directly from a measure algebra. These constructions are not particularly simple; in this book they take up Chapter 4 and the first half of Chapter 5. However, given some intuitive grasp of the nature of Riesz spaces, they are fairly straightforward, and provide invaluable insights.

For instance, another curiosity of traditional measure theory is the unimportance of any notion of homomorphism between measure spaces. This distinguishes it from all comparably abstract branches of mathematics. I believe that this deficiency occurs because the natural and important homomorphisms of the theory are between measure algebras and not between measure spaces at all. In §§45 and 54 I discuss such homomorphisms and their effects on the constructions I have set up.

The reader may be forgiven for wondering at this stage just how much he needs to know to cope with this book. A basic knowledge of functional analysis is essential. It would be possible to go a fair way with normed spaces alone; but this would shut off many of the most interesting ideas, and I have written on the assumption that I may call on the fundamental concepts of the theory of linear topological spaces. As for measure theory, in a formal sense I require none; I give every definition from that of measure space onwards. In an informal sense, I cannot pretend that this book is a genuine alternative to the traditional presentation of the elementary theory. §§61–3, while formally complete, are far too sketchy to be satisfactory as a first introduction. So if you are uncertain of your grounding in the subject, I refer you to the detailed advice in the Prerequisites section.

I have preferred, in ordering the material of this book, to arrange results by the contexts in which they apply; thus propositions which refer to arbitrary Riesz spaces go into §14 and those which refer only to Archimedean Riesz spaces go into §15. Now this is not, of course, a natural order for learning the subject. The beginner will certainly wish in the first instance to restrict his attention to Archimedean Riesz spaces, and leave any generalizations to one side for the moment. I think my decision is justified; it means that the hypotheses of successive theorems do not usually vary in arbitrary details, and are consequently much easier to recall precisely. But it makes a page-by-page progression inappropriate. To assist the reader to avoid spending too much time on material which will not be immediately essential, I have used asterisks to mark sections and propositions which can at first be passed over. They will be given references in the text when they come to be needed.

Most of this book is taken up with fulfilling the objects I described at the beginning of this preface. But I think that topological Riesz spaces are inherently fascinating, and I have laid Chapter 2 out as a survey of their elementary properties; and in Chapter 8 I try to take a special subject up to the area of present research work.

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Prerequisites

ZORN'S LEMMA, TRANSFINITE INDUCTION, AND THE AXIOM OF CHOICE The best introduction I know to this subject is in the appendix of KELLEY; HALMOS N.S.T. is also perfectly satisfactory. I shall apply the axiom of choice, and the principle of dependent choice for sequences, without comment, though when syntax allows I signal with such a word as 'choose' or 'choice'. Zorn's lemma is used most often in the form

> If $\mathscr{X}$ is any collection of sets such that for any subset $\mathscr{Y}$ of $\mathscr{X}$ which is totally ordered by ordinary inclusion, we have
> $$\bigcup \mathscr{Y} = \{t : \exists\ Y \in \mathscr{Y}, t \in Y\} \in \mathscr{X},$$
> then $\mathscr{X}$ has a maximal element.

(This is nearly what KELLEY calls the 'maximal principle'.) When employing this form, I shall generally say merely 'by Zorn's lemma, $\mathscr{X}$ has a maximal element', and leave it to the reader to verify that the hypotheses on $\mathscr{X}$ are satisfied – supposing, of course, that the verification is straightforward. But if the lemma is to be applied in the more general form

> If $\mathscr{X}$ is a partially ordered set such that every totally ordered subset has an upper bound in $\mathscr{X}$, then $\mathscr{X}$ has a maximal element,

then I shall give more details.

FUNCTIONAL ANALYSIS I can recommend BOURBAKI V, ROBERTSON & ROBERTSON, KELLEY & NAMIOKA, or SCHAEFER T.V.S.; any of these is adequate for the principal results. For details see §A1.

MEASURE THEORY As I explained in the Preface, I make no formal requirements in this field. But a knowledge of the elementary theory up to and including the Radon–Nikodým theorem would certainly be helpful. The first point is that although abstract measure theory is not particularly difficult in its early stages, the only easy examples of measure spaces are essentially trivial. There is no substitute for Lebesgue measure on the real line as a leading example. So at some stage the reader should make himself familiar with a proof that it exists.

Secondly, several of the most important ideas in this book appeared first in measure theory. Pre-eminent among these are the arguments leading up to the Radon–Nikodým theorem. I give this theorem in §63 with a proof which refers to almost every preceding chapter. In fact my proof is very closely related to one of the traditional ones, but its component ideas have been scattered. I think that it is illuminating to see how they can be joined more closely together in a direct proof. I shall give a concordance in the proper place.

Meanwhile, I do not want the reader to feel that he must take a course in measure theory before proceeding further. Perhaps the books suggested below should be regarded as a starred section; optional on first reading, but likely to be useful later.

R. G. Bartle, *The Elements of Integration* (Wiley, 1966)

H. L. Royden, *Real Analysis* (Macmillan, 1963)

H. Widom, *Lectures on Measure and Integration* (van Nostrand Reinhold, 1969)

J. H. Williamson, *Lebesgue Integration* (Holt, Rinehart & Winston, 1962).