1. Riesz spaces

The prerequisites for this chapter are few; an acquaintance with linear spaces is the principal requirement. A Riesz space is simply a linear space over the field $\mathbf{R}$ of real numbers which has a special kind of partial ordering, and all we need to know about partial orderings will be covered in §§ 11 and 13. But the theory of Riesz spaces is already rich, and some of the work in §§ 16 and 17 is far from trivial. It does, however, have to be taken seriously. These are the basic results which will enable us to handle Riesz spaces with assurance and facility.

11 Partially ordered sets

This section is little more than a list of definitions. As such I suggest that it should be read carefully once, together with the associated examples; but that there is no need to consciously memorize anything. You will find the index perfectly reliable.

Actually the concepts here have applications to every branch of mathematics, and they will mostly be familiar in everything but name. I think it is amusing and instructive to seek such applications out and consciously appreciate them.

Now to work:

11A Definition A **partially ordered set** is a pair $(A, \leqslant)$ where A is a set and $\leqslant$ is a binary relation on A such that:

(i) $a \leqslant a$ for every $a \in A$;

(ii) if a and b belong to A and $a \leqslant b$ and $b \leqslant a$, then $a = b$;

(iii) if a, b and c belong to A and $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

In this context we write $a \geqslant b$ for $b \leqslant a$ (that is, $\geqslant$ will always be the inverse relation to $\leqslant$), and $a < b$ or $b > a$ for $a \leqslant b$ and $a \neq b$.

For examples see 1XA and 1XC.

11B Suprema and infima Let $(A, \leqslant)$ be a partially ordered set, and B a subset of A. There may, or may not, be an $a \in A$ with the property

$$\text{for every} \quad c \in A, \quad a \leqslant c \quad \text{iff} \quad b \leqslant c \quad \text{for every} \quad b \in B,$$

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that is, $a \leqslant c$ iff c is an upper bound for B. Clearly there can be at most one such a [11A(ii)]. If a has this property, we say that 'B has a least upper bound, which is a', or '$\sup B$ exists, and $\sup B = a$', or simply '$\sup B = a$'.

Similarly, $\inf B$, if it exists, is that one element of A such that

$$\text{for every} \quad c \in A, \quad c \leqslant \inf B \quad \text{iff} \quad c \leqslant b \quad \text{for every} \quad b \in B.$$

We observe that

$$\inf B = \sup\{a : a \leqslant b \ \forall \ b \in B\},$$

$$\sup B = \inf\{a : b \leqslant a \ \forall \ b \in B\},$$

$$\inf A = \sup \varnothing, \quad \sup A = \inf \varnothing,$$

all these inequalities being true in the sense that if one side is defined, so is the other, and they are then equal.

For a warning of the dangers in this notation, see 1XB.

11C Directed sets Let $(A, \leqslant)$ be a partially ordered set. A set $B \subseteq A$ is **directed upwards,** $B\uparrow$, if for every pair a, b of elements of B there is a $c \in B$ such that $a \leqslant c$ and $b \leqslant c$.

I shall write $B \uparrow a$ to mean that $B\uparrow$ and that $\sup B = a$. Observe, for instance, that $\{a\} \uparrow a$ for every $a \in A$.

Associated with a directed set is a filter. If B is a non-empty set which is directed upwards, I shall write $\mathscr{F}(B\uparrow)$ for the filter on A with base

$$\{\{a : a \in B, a \geqslant b\} : b \in B\}.$$

$\mathscr{F}(B\uparrow)$ is sometimes called the **filter of sections** of the directed set B.

$B\downarrow$ ('B is directed downwards'), $B \downarrow a$ and $\mathscr{F}(B\downarrow)$ are defined in the same way, but upside down.

11D Functions Let $(A, \leqslant)$ and $(B, \preccurlyeq)$ be partially ordered sets. A function $f: A \to B$ is **increasing** if $a \leqslant b \Rightarrow fa \preccurlyeq fb$. When f is a sequence $\langle a_n \rangle_{n \in \mathbf{N}}$ say, I shall write $\langle a_n \rangle_{n \in \mathbf{N}}\uparrow$ to mean that it is an increasing sequence, that is, that $a_n \preccurlyeq a_{n+1}$ for every $n \in \mathbf{N}$. Similarly, $\langle a_n \rangle_{n \in \mathbf{N}}$ is a **decreasing** sequence, $\langle a_n \rangle_{n \in \mathbf{N}}\downarrow$, if $a_{n+1} \preccurlyeq a_n$ for every $n \in \mathbf{N}$. By analogy with 11C, I shall write $\langle a_n \rangle_{n \in \mathbf{N}} \uparrow a$ to mean that $\langle a_n \rangle_{n \in \mathbf{N}}\uparrow$ and that $\sup_{n \in \mathbf{N}} a_n = a$. $\langle a_n \rangle_{n \in \mathbf{N}} \downarrow a$ is defined similarly.

An increasing function $f: A \to B$ is **order-continuous** if

$$\sup f[C] = fa \quad \text{whenever } C \uparrow a \text{ and } C \text{ is not empty}$$

$$\inf f[C] = fa \quad \text{whenever } C \downarrow a \text{ and } C \text{ is not empty.}$$

(Of course, if f is increasing, then $f[C]$ is directed whenever C is.) Note that it is only directed sets which must have their sups and infs preserved in this way.

*There are two more concepts which will be useful to us in special circumstances. An increasing function $f: A \to B$ is **order-continuous on the left** if it satisfies the first half of the condition for order-continuity, that is, if $f[C] \uparrow fa$ whenever $C \uparrow a$ and $C \neq \varnothing$. And f is **sequentially order-continuous** if

$$\langle fa_n \rangle_{n \in \mathbf{N}} \uparrow fa \quad \text{whenever} \quad \langle a_n \rangle_{n \in \mathbf{N}} \uparrow a$$

and

$$\langle fb_n \rangle_{n \in \mathbf{N}} \downarrow fb \quad \text{whenever} \quad \langle b_n \rangle_{n \in \mathbf{N}} \downarrow b.$$

***11E Order-closed sets** If $(A, \leqslant)$ is a partially ordered set and $B \subseteq A$, I shall write

$$\mathscr{I}B = \{a: \exists\ C \subseteq B, \quad C \neq \varnothing, \quad C \uparrow a \quad \text{in} \quad A\},$$

$$\mathscr{D}B = \{a: \exists\ C \subseteq B, \quad C \neq \varnothing, \quad C \downarrow a \quad \text{in} \quad A\}.$$

Then $B \subseteq \mathscr{I}B$ and $B \subseteq \mathscr{D}B$.

B is **order-closed** if $\mathscr{I}B = B = \mathscr{D}B$.

***11F Order-bounded sets** Let $(A, \leqslant)$ be a partially ordered set. A set $B \subseteq A$ is **order-bounded** if it has both upper and lower bounds in A, i.e. if $B \subseteq [b, c] = \{a: b \leqslant a \leqslant c\}$ for some b and c in A.

***11G Products** Let $\langle (A_\iota, \leqslant_\iota) \rangle_{\iota \in I}$ be an indexed family of partially ordered sets. Let A be the cartesian product $\prod_{\iota \in I} A_\iota$, and define a relation $\leqslant$ on A by $a \leqslant b$ iff $a(\iota) \leqslant b(\iota)$ for every $\iota \in I$. Then $(A, \leqslant)$ is a partially ordered set, the **product** of the family $\langle (A_\iota, \leqslant_\iota) \rangle_{\iota \in I}$. If, for each $\iota \in I$, $\pi_\iota: A \to A_\iota$ is the canonical map, then π_ι is increasing and order-continuous.

***11H Exercises** (a) Let $(A, \leqslant)$ be a partially ordered set. Then there is a topology on A for which the closed sets are precisely the order-closed sets of A. [See 1XC.]

(b) Let $(A, \leqslant)$ and $(B, \leqslant)$ be partially ordered sets. Then an increasing function $f: A \to B$ is order-continuous iff $f^{-1}[C]$ is order-closed in A for every order-closed $C \subseteq B$.

Notes and comments We shall see a greatest lower bound or a least upper bound on every other page of this book; so it will be as well to grasp firmly the formal definitions in 11B. I generally take these formalities seriously; I want to know exactly what I mean by such expressions as inf $\varnothing$.

I cannot emphasize too strongly the fact that the least upper bound of a set depends on the partially ordered set in which it lies. There would be formal advantages in writing

$$\sup_{(A,\leqslant)} B$$

to mean 'the least upper bound of B taken in the partially ordered set $(A, \leqslant)$'. Of course this would be absurdly cumbersome; but the abbreviation 'sup B' must always be recognized as such. In 1XB I give a simple example of the danger. There's a more elaborate one in 4XF, and the distinctions become very important in Chapter 7. The same warning, of course, applies to the notations $\mathscr{I}$ and $\mathscr{D}$ in 11E.

There is a perfect symmetry in the notion of partially ordered set; if $\leqslant$ is a partial ordering, so is its inverse $\geqslant$. Consequently all the associated definitions are doubled, as in 11B and 11C. I want to call attention to the concept of the filter of sections $\mathscr{F}(B\uparrow)$, defined in 11C. I think that this filter is almost the most important thing associated with a directed set. You may have seen it already in the correspondence between net-convergence and filter-convergence in topological spaces [KELLEY 2L]. Note the technical point that $\mathscr{F}(B\uparrow)$ is a filter on the whole space A, not on B.

For another look at the material of this section, see BOURBAKI I, chapter III, § 1.

12 Partially ordered linear spaces

In this short section I discuss the simplest way in which a linear space structure and a partial ordering can be related. Although we shall very rarely have any reason to consider partially ordered linear spaces which are not Riesz spaces [§ 14], I think that the results here are clarified by being placed in their natural context.

Here, and everywhere in this book, all linear spaces have the real numbers for their underlying field. There do exist applications in which it is more convenient to have the complex numbers; but I think that these are best approached by way of the real case. Other fields are so far merely curiosities from the standpoint of this theory.

12A Definition A **partially ordered linear space** is a quadruple $(E, +, ., \leqslant)$ where $(E, +, .)$ is a linear space over the field $\mathbf{R}$ of real numbers and $\leqslant$ is a partial ordering on E such that

(i) if $x \leqslant y$, then $x+z \leqslant y+z$ for every $z \in E$;

(ii) If $x \geqslant 0$ in E, then $\alpha x \geqslant 0$ whenever $\alpha \geqslant 0$ in $\mathbf{R}$.

12B Positive cones From 12A(i), we see that $x \leqslant y \Leftrightarrow 0 \leqslant y-x$. So $\leqslant$ is determined entirely by $E^+ = \{x: x \in E, x \geqslant 0\}$, the **positive cone** of E. Given a linear space E over $\mathbf{R}$ and a set $P \subseteq E$, there is a partial ordering $\leqslant$ on E such that $(E, \leqslant)$ is a partially ordered linear space and $P = E^+$ iff

$$P \cap (-P) = \{0\} \quad \text{[for 11A (i) and (ii)]},$$

$$P + P \subseteq P \quad \text{[for 11A (iii)]},$$

$$\alpha P \subseteq P \quad \forall\ \alpha \geqslant 0 \quad \text{[for 12A (ii)]}.$$

In particular, $P = \{0\}$ will do [cf. 1XA].

12C Lemma Let E be a partially ordered linear space, $x \in E$, A, $B \subseteq E$. Then

(a) $\sup(x+A) = x + \sup A$ if either side exists.

(b) $\sup(-A) = -\inf A$ if either side exists.

(c) $\sup(A+B) = \sup A + \sup B$ if the right-hand side exists.

(d) If $\alpha \geqslant 0$, $\sup(\alpha A) = \alpha \sup A$ if the right-hand side exists.

Proof of (c) Apply (*a*) twice, as follows:

$$\begin{aligned}\sup A + \sup B &= \sup(A + \sup B)\\ &= \sup\{x + \sup B : x \in A\}\\ &= \sup\{\sup\{x+y : y \in B\} : x \in A\}\\ &= \sup\{x+y : x \in A, y \in B\}\\ &= \sup(A+B).\end{aligned}$$

Notes and comments The correspondence between an ordering and a positive cone [12B] is extremely important; it is one of the easiest ways of defining partial orderings on linear spaces.

In 12C I prove only (c), because the other parts are direct consequences of the definitions. In a partially ordered linear space E, the map $y \mapsto x+y : E \to E$ is an order-automorphism for every $x \in E$, and therefore must preserve suprema and infima. The same applies to the map $y \mapsto \alpha y : \boldsymbol{E} \to \boldsymbol{E}$ for every $\alpha > 0$. On the other hand, the map $y \mapsto -y : \boldsymbol{E} \to \boldsymbol{E}$ is order-reversing, that is, $x \leqslant y \Leftrightarrow -y \leqslant -x$, so it exchanges suprema for infima.

Although I shall have no space to discuss them, many of the ideas of this book, set out for Riesz spaces, have extensions to partially ordered linear spaces. Compatible topologies [§ 21] are an obvious example; Lebesgue topologies [§ 24] are another. Some of these developments may be found in PERESSINI.

13 Lattices

Once again we must have a section consisting mostly of definitions. I maintain a careful separation between general remarks on lattices and those concerning linear spaces because I wish to apply the former to Boolean lattices in § 41.

Although it is possible to regard a lattice as a set with two binary algebraic operations, governed by certain identities, I prefer to take it as a special kind of partially ordered set. This makes it easier to think of the suprema and infima of infinite sets, which is something we must do continually.

The important sections below are A–E, though G should be examined because it attaches a rather unusual meaning to the word 'disjoint'.

13A Definitions A **lattice** is a partially ordered set $(A, \leqslant)$ such that $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all elements a and b of A. It follows at once (by induction on the number of elements in B) that $\sup B$ and $\inf B$ exist for every non-empty finite set $B \subseteq A$.

In a lattice A, we write $a \vee b$ and $a \wedge b$ for $\sup\{a, b\}$ and $\inf\{a, b\}$ respectively. Thus $\vee$ and $\wedge$ may be thought of as binary operations on A, the **lattice operations.** Since

$$a \leqslant b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a,$$

the lattice can be defined in terms of these operations. But apart from a few special instances [e.g. 22F below], I think it is easier to regard the partial ordering as fundamental. From this point of view, for instance, the identity.

$$(a \vee b) \vee c = a \vee (b \vee c)$$

is trivial, as both sides are equal to $\sup\{a, b, c\}$.

13B Dedekind completeness (a) A lattice A is **Dedekind complete,** or **order-complete,** if every non-empty $B \subseteq A$ which has an upper bound in A has a least upper bound.

(b) A lattice A is **Dedekind σ-complete,** or **sequentially Dedekind complete,** if every non-empty countable $B \subseteq A$ which has an upper bound or a lower bound has a least upper bound or greatest lower bound respectively.

13C Notes These definitions are extremely important. Observe:

(a) If A is a lattice, then A is Dedekind complete iff every non-empty subset of A which has a lower bound has a greatest lower bound. **P** Suppose that A is Dedekind complete and that $B \subseteq A$ has a lower bound and is not empty. Let

$$C = \{a : a \in A, \ a \text{ is a lower bound for } B\}.$$

Then C is a non-empty subset of A with an upper bound, so $\sup C$ exists. But $\sup C = \inf B$. The reverse implication is proved in exactly the same way. **Q**.

Consequently a Dedekind complete lattice is indeed Dedekind σ-complete. The definition of Dedekind σ-completeness must be given in terms of both suprema and infima because there is no way to get a countable C from a countable B.

(b) Let A be any lattice, and $B \subseteq A$. Let

$$B_1 = \{\sup C : C \subseteq B, \ C \text{ is finite and not empty}\}.$$

Then B_1 is closed under the lattice operation $\vee$ and has the same upper bounds as B. So 13Ba could be rewritten as: 'a lattice A is Dedekind complete if every non-empty set $B \subseteq A$, which is directed upwards and has an upper bound, has a least upper bound'. This will often be easier to prove [e.g. 16Db].

13D Distributive lattices A lattice A is **distributive** if

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c), \quad (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

for all a, b, $c \in A$; that is, if each of $\vee$ and $\wedge$ is distributive over the other.

All the lattices considered in this book are distributive, and I shall make no attempt to discuss any other kind. BIRKHOFF is still an excellent introduction to general lattice theory.

Exercise Show that each of the two halves of the condition above implies the other, so that either would be sufficient for the definition.

13E Lattice homomorphisms If A and B are lattices, a function $f: A \to B$ is a **lattice homomorphism** if

$$f(a \vee b) = fa \vee fb, \quad f(a \wedge b) = fa \wedge fb$$

for all $a, b \in A$. A lattice homomorphism must be increasing. Observe that an order-continuous lattice homomorphism [11D] is precisely a function which preserves all suprema and infima of non-empty sets; to see this we have to use the trick of 13Cb to transform an arbitrary supremum or infimum into the supremum or infimum of a directed set, so as to apply the hypothesis of order-continuity.

13F Sublattices Let A be a lattice. A **sublattice** of A is a subset of A closed under $\vee$ and $\wedge$. *A **σ-sublattice** is a sublattice B of A such that if $C \subseteq B$ is countable and not empty, and $\inf C$ or $\sup C$ exists in A, then it belongs to B.

13G Disjoint sets and functions Let A be a lattice with a least member a_0. I shall call a set $B \subseteq A$ **disjoint** (in A) if $b \wedge c = a_0$ whenever b and c are distinct members of B. (Observe that possibly $a_0 \in B$.) If I is any set, a function $f: I \to A$ is **disjoint** if $f\iota \wedge f\kappa = a_0$ whenever ι and κ are distinct members of I.

*I shall often use the following phrase: 'C is disjoint in E^+', where E is a Riesz space and E^+ is its positive cone. Now E^+ is a sublattice of the lattice E, and has a least member, viz. 0; so what I mean is that $x \wedge y = 0$ whenever x and y are distinct members of C.

***13H Products** Let $\langle (A_\iota, \leqslant_\iota) \rangle_{\iota \in I}$ be a family of partially ordered sets, and suppose that none of the A_ι is empty. Let $\leqslant$ be the product partial ordering defined on $A = \prod_{\iota \in I} A_\iota$ [11G]. Then

(a) $(A, \leqslant)$ is a lattice iff every $(A_\iota, \leqslant_\iota)$ is.

(b) $(A, \leqslant)$ is distributive, or Dedekind complete, or Dedekind σ-complete, iff every $(A_\iota, \leqslant_\iota)$ is.

(c) The canonical maps from A to each A_ι are always lattice homomorphisms.

Notes and comments As I have rigorously expunged from this section everything which will not be used later, it is not an adequate introduction to lattice theory. The only two kinds of lattice which we shall be thinking about, Riesz spaces and Boolean lattices, share many special properties; to begin with, they are distributive, which makes them rather uninteresting as lattices. The distributive property is so natural that one is liable to use it unconsciously; which is dangerous if there are any nondistributive lattices about. But I think that for now we can accept these simplifications cheerfully.

Since most of the objects of discussion of this book are lattices, I shall not trouble with many examples at this point. The obvious ones to glance at, before continuing to the next section, are 1XB, 1XC and 4XA.

14 Riesz spaces

As soon as we join §§ 12 and 13 together, and consider partially ordered linear spaces which are lattices, the character of our study changes completely. Riesz spaces have a remarkable wealth of properties.

Some of the most elementary, identities relating the lattice and linear space operations, are in 14B. These are so numerous that one naturally seeks an alternative to memorization; and such an alternative exists in the '**R**-test', which runs 'if an identity is true in **R**, it is true in all Riesz spaces'. Actually, there is a perfectly valid metatheorem along these lines. But the metatheorem is a good deal more complex then the work it eliminates; besides, direct proofs are more illuminating of associated structures such as lattice-ordered groups. So I give the identities most commonly required in an order which makes each an easy consequence of the preceding ones.

The really surprising thing about Riesz spaces is the strong distributive law which they obey [14D]. After this, the rest of the section is fairly straightforward, with the expected relationships between products, subspaces, quotients and homomorphisms.

14A Definitions A **Riesz space,** or **vector lattice,** is a partially ordered linear space $(\boldsymbol{E}, +, ., \leqslant)$ such that $(\boldsymbol{E}, \leqslant)$ is a lattice.

If $\boldsymbol{E}$ is a Riesz space, we write

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x \vee (-x)$$

for any $x \in \boldsymbol{E}$.

Exercise Let $\boldsymbol{E}$ be a partially ordered linear space such that $\sup\{x, 0\}$ exists for every $x \in \boldsymbol{E}$. Then $\boldsymbol{E}$ is a Riesz space. [Use 12Ca and 12Cb.]

14B Elementary identities Let E be a Riesz space, x, y and z members of E, and α and β real numbers. Then

(a) $(x \vee y)+z = (x+z) \vee (y+z)$ [special case of 12Ca].

(b) $(x \wedge y)+z = (x+z) \wedge (y+z)$ [similarly].

(c) $-(x \wedge y) = (-x) \vee (-y)$ [special case of 12Cb].

(d) If $\alpha \geqslant 0$, $\alpha x \vee \alpha y = \alpha(x \vee y)$ [special case of 12Cd].

(e) If $\alpha \geqslant 0$, $\alpha x \wedge \alpha y = \alpha(x \wedge y)$ [similarly].

(f) $2(x \vee y) = 2x \vee 2y = x+y+|x-y|$ [definition of $|\ |$ and (a) above].

(g) $2(x \wedge y) = x+y-|x-y|$ [similarly, using (b) and (c)].

(h) $x+y = x \vee y + x \wedge y$ [adding (f) and (g)].

(i) $x = x^+ - x^-$ [apply (a) to $x+x^-$].

(j) $|x| = x^+ + x^-$ [apply (d) and (a) to $x+2x^-$].

(k) $x^+ \vee x^- = |x|$ [from the definitions, since $|x| \geqslant 0$ by (j)].

(l) $x^+ \wedge x^- = 0$ [by (j), (k) and (h)].

(m) $|\alpha x| = |\alpha|\,|x|$ [consider $\alpha \geqslant 0$ and $\alpha \leqslant 0$ separately].

(n) $|x+y| \leqslant |x|+|y|$ [since $x+y \leqslant x+|y| \leqslant |x|+|y|$; and similarly $-x-y \leqslant |x|+|y|$].

14C Definitions Let E be a Riesz space. A set $A \subseteq E$ is **solid** if $y \in A$ whenever there is an $x \in A$ such that $|y| \leqslant |x|$. The intersection of any set of solid sets is solid.

If A is any subset of E, the set $\{y : \exists\, x \in A, |y| \leqslant |x|\}$ is solid; it is the smallest solid set including A, and is called the **solid hull** of A.

14D The distributive law Let E be a Riesz space and A a subset of E such that $\sup A$ exists. Then, for any $x \in E$, $\sup\{x \wedge y : y \in A\}$ exists and is equal to $x \wedge \sup A$.

Proof Plainly $x \wedge \sup A$ is an upper bound for $\{x \wedge y : y \in A\}$. Conversely, suppose that $z \geqslant x \wedge y$ for every $y \in A$. Then

$$\begin{aligned} z &\geqslant (x-y) \wedge 0 + y \\ &\geqslant (x-\sup A) \wedge 0 + y, \quad \text{for every } y \in A. \end{aligned}$$

So

$$z \geqslant (x-\sup A) \wedge 0 + \sup A = x \wedge \sup A.$$

Similarly $x \vee \inf A = \inf\{x \vee y : y \in A\}$ if the left-hand side exists.