INTRODUCTION

The title *Differential Analysis* indicates clearly the content of this book. It is concerned with those parts of analysis in which the idea of differentiation, derivative or differential plays a central role. It is true that the functions to be discussed all take values in normed spaces and, for a large part of the book, they also have subsets of normed spaces as domains, but the basic aim is the generalization and subsequent application of the fundamental theorems of the differential calculus of functions of one real variable.

To be sure, this generalization, simple as it is in essence, extends the range of applications widely. One differential equation involving functions of one variable but with values in a normed space can be equivalent to an infinite system of equations involving scalar-valued functions. The calculus of variations becomes a part of the 'ordinary' theory of maxima and minima. Very general questions about constrained maxima and minima become problems about inclusions between tangent spaces. When set in Banach space, the Newton method, giving an iterative procedure for finding (approximate) solutions of equations, becomes applicable to integral and differential equations.

Although the theory has such a wide import, it remains basically elementary. The reader who glances through the appendix (listing results required in the text) will disbelieve this assertion: some deep theorems are given there. This is indeed true, but except in one or two sections, appeals to these results are rare. For example, the Ascoli–Arzelà theorem is used (on a few occasions) to obtain the existence of solutions to the fundamental equation $y' = f(t, y)$, but the theory then proceeds without further recourse to sophisticated theorems. The book has the flavour of the calculus, not the flavour of functional analysis.

For functions of one real variable with values in a normed space, the definition of derivative is straightforward enough (though the formulation and proof of results often requires ingenuity). This forms the subject of Chapter 1, while the applications to the equation $y' = f(t, y)$ are dealt with in Chapter 2. For functions whose domains are subsets of normed spaces, the 'correct' definition of differential is less clear, and there are several possibilities. One of the best established, and the strongest, is the Fréchet differential, which is discussed in Chapter 3. Two of the other candidates are presented in Chapter 4. The Gâteaux (or directional) differential is the weakest: it can be defined in any vector space. The Hadamard differential lies between the other two; it coincides with the Fréchet differential in finite dimensions, but is more general in infinite-dimensional spaces. It turns out to be exactly what is required for certain considerations in tangent spaces and in aspects of the theory of differential equations (for example, 'differentiation along the curve').

The above discussion has mentioned functions of a real variable. In fact, many of the theorems in the book are true for functions of a complex variable, or when the field of scalars of the normed space is complex. However, to leave the matter there would be misleading. Just as in the case of one variable, there are significant differences between the real and complex theories. No attempt has been made to obtain the stronger conclusions valid for the latter case, although the results have been stated for both real and complex scalars where this was possible.

There will also be found in the book three long historical notes. These are intended to put the modern theory presented in an historical context. They do not, of course, amount to a complete history of the subject, but it is hoped they will be seen not merely as curiosities, but as a real aid in understanding the material.

Each result in the book is assigned a triple of numbers, for example, (3.6.2); this means the second numbered result of §3.6, the sixth section of Chapter 3. Within some sections certain formulae are numbered on the right, for example, (6); a reference to (6) means the sixth numbered formula in the section in which the reference occurs, unless other instructions are given.

1

Differentiation of functions of one real variable

Throughout this chapter, Y denotes a normed vector space, and, except where noted, Y may be over either the real or complex field. The completeness of Y rarely enters our arguments, and we therefore allow Y to be incomplete unless otherwise stated.

1.1 The derivative of a real- or vector-valued function of a real variable

Let $A \subseteq \mathbf{R}$, let $\phi : A \to Y$, and let t_0 be a non-isolated point of A, i.e. $t_0 \in A$ and for each $\delta > 0$ there exists $t \in A$ satisfying $0 < |t - t_0| < \delta$.† The function ϕ is said to have a derivative at t_0 if the limit ‡

$$\lim_{t \to t_0} \frac{\phi(t) - \phi(t_0)}{t - t_0} = \lim_{h \to 0} \frac{\phi(t_0 + h) - \phi(t_0)}{h}$$

exists in Y, and the value of this limit is then called the *derivative of* ϕ *at* t_0, and is denoted by $\phi'(t_0)$. The *derivative* ϕ' of ϕ is the function $t \mapsto \phi'(t)$ whose domain is the set of non-isolated points t of A at which $\phi'(t)$ exists (ϕ' may, of course, be the empty function).

In the sequel, when we say that a function ϕ has a derivative at a point t_0, we take it as understood that t_0 is a non-isolated point of its domain. The most important cases are those where the domain of ϕ is an interval (when every point of the domain is non-isolated) and where t_0 is an interior point of the domain.*

† t_0 is a point of A that is also a point of accumulation of A.

‡ Here, by 'the limit as t tends to t_0', we mean that t is to tend to t_0 through the points of the set $A \backslash \{t_0\}$, so that the only values of t involved are ones for which the quotient $(\phi(t) - \phi(t_0))/(t - t_0)$ is defined. This convention is followed throughout the book.

* It is probably more usual to define the derivative of a function ϕ only at an interior point of its domain. However, in the theory of differential equations we frequently encounter functions from a compact interval J into Y possessing a derivative at each point of J. The endpoints of J can, of course, be dealt with by the use of left-hand and right-hand derivatives (see below), but this leads to a clumsy notation, and the wider definition of derivative adopted here seems preferable.

A function ϕ which possesses a derivative $\phi'(t)$ at a point t is sometimes said to be *differentiable at* t. Similarly, if $\phi'(t)$ exists at every point t of an interval I, then ϕ is said to be *differentiable on* I; if in addition ϕ' is continuous on I then ϕ is *continuously differentiable on* I.

When the function ϕ is given by a specific formula, the derivative $\phi'(t_0)$ of ϕ at t_0 is sometimes denoted by $[d\phi(t)/dt]_{t_0}$, so that, for example, $[dt^2/dt]_{t_0}$ denotes the derivative of the function $t \mapsto t^2$ at t_0. We also write $d\phi(t)/dt$ for the value of the derivative of ϕ at the point t.

A number of other notations for the derivative are in use; the most common, dy/dt, employs a 'dependent variable' y, which stands either for the function ϕ or for the value of ϕ at t, so that dy/dt can be interpreted as either the function ϕ' or its value $\phi'(t)$ according to context. We use this 'dependent variable' notation in a purely formal way in Chapter 2, and otherwise do not use any of these further notations.

Example (a)

If $\phi : A \to Y$ is constant, then $\phi'(t)$ exists and is equal to 0 at each non-isolated point t of A.

Example (b)

If $\phi : \mathbf{R} \to Y$ is given by

$$\phi(t) = c_0 + tc_1 + t^2c_2 + \dots + t^m c_m,$$

where $c_0, \dots, c_m \in Y$, then for all $t \in \mathbf{R}$

$$\phi'(t) = c_1 + 2tc_2 + \dots + mt^{m-1}c_m.$$

Example (c)

Let $\phi : A \to \mathbf{R}^m$, where $A \subseteq \mathbf{R}$, let $\phi_1, \dots, \phi_m : A \to \mathbf{R}$ be the components of ϕ given by

$$\phi(t) = (\phi_1(t), \dots, \phi_m(t)) \qquad (t \in A), \tag{1}$$

and let t_0 be a non-isolated point of A. Then $\phi'(t_0)$ exists and is equal to $d = (d_1, \dots, d_m)$ if and only if, for $i = 1, \dots, m$, $\phi_i'(t_0)$ exists and is equal to d_i. This follows immediately from the identity

$$\left\| \frac{\phi(t) - \phi(t_0)}{t - t_0} - d \right\| = \left(\sum_{i=1}^{m} \left| \frac{\phi_i(t) - \phi_i(t_0)}{t - t_0} - d_i \right|^2 \right)^{1/2}.$$

For example, if $\phi : \mathbf{R} \to \mathbf{R}^3$ is given by $\phi(t) = (\cos t, \sin t, t)$, then, for all $t \in \mathbf{R}$, $\phi'(t) = (-\sin t, \cos t, 1)$.

More generally, we may replace $\mathbf{R}^m$ by the product of m normed vector spaces $Y_1 \times Y_2 \times \ldots \times Y_m$, the components $\phi_i : A \to Y_i$ $(i = 1, \ldots, m)$ being defined by (1) as before.

Example (d)

Let Y be a complex Hilbert space, let $\phi : A \to Y$, where $A \subseteq \mathbf{R}$, and let $\psi(t) = \|\phi(t)\|^2$. Then $\psi'(t) = 2\,\mathrm{Re}\,\langle \phi'(t), \phi(t) \rangle$ whenever $\phi'(t)$ exists. In fact

$$\begin{aligned}\psi(t+h) - \psi(t) &= \|\phi(t+h)\|^2 - \|\phi(t)\|^2 \\ &= \langle \phi(t+h), \phi(t+h) \rangle - \langle \phi(t), \phi(t) \rangle \\ &= \langle \phi(t+h) - \phi(t), \phi(t+h) \rangle + \langle \phi(t), \phi(t+h) - \phi(t) \rangle,\end{aligned}$$

whence

$$\begin{aligned}\psi'(t) &= \langle \phi'(t), \phi(t) \rangle + \langle \phi(t), \phi'(t) \rangle \\ &= \langle \phi'(t), \phi(t) \rangle + \overline{\langle \phi'(t), \phi(t) \rangle} \\ &= 2\,\mathrm{Re}\,\langle \phi'(t), \phi(t) \rangle.\end{aligned}$$

If Y is a real Hilbert space, then $\psi'(t) = 2\langle \phi'(t), \phi(t) \rangle$.

Example (e)

Let c_0 denote the Banach space of sequences $y = (y_n)_{n \geq 1}$ of real numbers that converge to the limit 0, with the sup norm $\|y\| = \sup_n |y_n|$. Let also ϕ be a function from a set $A \subseteq \mathbf{R}$ into c_0, and for $n = 1, 2, \ldots$ define $\phi_n : A \to \mathbf{R}$ by $\phi(t) = (\phi_n(t))$ (so that the sequence $(\phi_n(t))$ is convergent to the limit 0 for each $t \in A$). If $\phi'(t_0)$ exists for some non-isolated point t_0 of A, then $\phi_n'(t_0)$ exists for each n and $\phi'(t_0) = (\phi_n'(t_0))$, so that also $(\phi_n'(t_0)) \in c_0$. (For, if $\phi'(t_0) = (d_n)$, then for all $n \geq 1$ and all $t \in A$

$$\left| \frac{\phi_n(t) - \phi_n(t_0)}{t - t_0} - d_n \right| \leq \left\| \frac{\phi(t) - \phi(t_0)}{t - t_0} - \phi'(t_0) \right\|.)$$

The converse is false, i.e. there is a function ϕ such that $\phi'(t_0)$ does not exist, even though $\phi_n'(t_0)$ exists for each n and $(\phi_n'(t_0)) \in c_0$. For example, let $A = \mathbf{R}$, let $\phi_n(t) = n^{-1} \log(1 + n^2 t^2)$ $(n \geq 1)$, and let $\phi(t) = (\phi_n(t))$. If $\phi'(0)$ exists, then, by the preceding argument, it must be equal to $(\phi_n'(0)) = (0)$. However, for all $t \neq 0$

$$\left\| \frac{\phi(t) - \phi(0)}{t - 0} - (0) \right\| = \left\| \frac{\phi(t)}{t} \right\| = \sup_n \left(|nt|^{-1} \log(1 + n^2 t^2) \right),$$

and the expression under the supremum sign on the right here is greater than or equal to log 2 when t is the reciprocal of an integer, so that $\phi'(0)$ does not exist.

Similar results hold for other sequence spaces.

Example (f)

A derivative need not be continuous, even if it exists at each point of an interval. Consider for instance the function $\phi:\mathbf{R}\to\mathbf{R}$ given by

$$\phi(t)=t^2\sin(\pi/t^2)\quad(t\neq 0),\quad \phi(0)=0.$$

Here $\phi'(0)=0$, and for $t\neq 0$

$$\phi'(t)=2t\sin(\pi/t^2)-(2\pi/t)\cos(\pi/t^2),$$

so that $\phi'(1/n^{1/2})=\pm 2\pi n^{1/2}$ according as n is an odd or even integer (thus the derivative is in fact unbounded on each compact interval containing 0).†

The right-hand and left-hand derivatives $\phi'_+(t_0)$ and $\phi'_-(t_0)$ are defined in a manner similar to that for $\phi'(t_0)$. If t_0 is a point of A such that for each $\delta>0$ there exists $t\in A$ satisfying $t_0<t<t_0+\delta$, then the (*strong*) *right-hand derivative* $\phi'_+(t_0)$ of ϕ at t_0 is the limit

$$\lim_{t\to t_0+}\frac{\phi(t)-\phi(t_0)}{t-t_0}=\lim_{h\to 0+}\frac{\phi(t_0+h)-\phi(t_0)}{h}$$

whenever this limit exists in Y. The *left-hand derivative* $\phi'_-(t_0)$ is defined similarly.

Example (g)

Let I be an interval in $\mathbf{R}$, and let $\phi:I\to\mathbf{R}$ be convex. Then $\phi'_+(t)$ and $\phi'_-(t)$ exist and satisfy $\phi'_-(t)\leq\phi'_+(t)$ for each interior point t of I (so that also ϕ is continuous on I°), and $\phi'_-(t)=\phi'_+(t)$ for all except a countable set of $t\in\,]a,b[$. Further, ϕ'_- and ϕ'_+ are both increasing on I°.

Let s, t, u be points of I° such that $s<u<t$. We see from (A.3.6) that

$$\frac{\phi(u)-\phi(s)}{u-s}\leq\frac{\phi(t)-\phi(s)}{t-s}\leq\frac{\phi(t)-\phi(u)}{t-u}.$$

The left-hand inequality, with s,u,t replaced first by $s,s+h,s+k$, and then by $s,s+h,t$, implies that the function $h\mapsto(\phi(s+h)-\phi(s))/h$ is increasing on $]0,t-s]$ and is bounded above there by $(\phi(t)-\phi(s))/(t-s)$.

† The expressions $\cos(\pi/t^m)$ and $\sin(\pi/t^m)$ are frequently useful in the construction of counter-examples, and various instances will be encountered later.

Similarly, if $s-k\in I^{\circ}$, where $k>0$, then the right-hand inequality, with s,u,t replaced by $s-k,s,s+h$, implies that $h\mapsto(\phi(s+h)-\phi(s))/h$ is bounded below by $(\phi(s)-\phi(s-k))/k$, and hence $\phi'_{+}(s)$ exists and satisfies

$$\frac{\phi(s-k)-\phi(s)}{k}\le\phi'_{+}(s)\le\frac{\phi(t)-\phi(s)}{t-s}.$$

In the same way, we show that $\phi'_{-}(t)$ exists and satisfies

$$\frac{\phi(t)-\phi(s)}{t-s}\le\phi'_{-}(t)\le\frac{\phi(t+l)-\phi(t)}{l},$$

where $l>0$ and $t+l\in I^{\circ}$, and on combining these results we obtain that

$$\phi'_{-}(s)\le\phi'_{+}(s)\le\frac{\phi(t)-\phi(s)}{t-s}\le\phi'_{-}(t)\le\phi'_{+}(t).$$

Finally, let E be the set of $t\in I^{\circ}$ for which $\phi'_{-}(t)<\phi'_{+}(t)$, and for each $t\in E$ let $J_t=]\phi'_{-}(t),\phi'_{+}(t)[$. By the central pair of inequalities above, $J_s\cap J_t=\varnothing$ for all $s,t\in E$. It therefore follows by a standard argument that the set of intervals J_t is countable, whence so also is E.

In the theorems of this section we state the results for the ordinary derivative ϕ', and take for granted their extensions to $\phi'_{\pm}$.

The following result gives some equivalent formulations of the definition of the derivative; the proof is immediate.

(1.1.1) *Let $\phi:A\to Y$, where $A\subseteq\mathbf{R}$, and let t_0 be a non-isolated point of A. Then the following statements are equivalent:*

(i) *ϕ has a derivative at t_0 equal to d;*

(ii) *the function $\lambda:A\to Y$ given by*

$$\lambda(t)=\frac{\phi(t)-\phi(t_0)}{t-t_0}\quad(t\in A,t\neq t_0),\quad\lambda(t_0)=d,$$

is continuous at t_0;

(iii) *for all $t\in A$*

$$\phi(t)=\phi(t_0)+(t-t_0)d+(t-t_0)v(t),$$

where $v:A\to Y$ is continuous at t_0 and has the value 0 there;

(iv) $$(\phi(t_0+h)-\phi(t_0)-hd)/|h|\to 0 \textit{ in } Y \textit{ as } h\to 0.$$

The existence of $\phi'(t_0)$ obviously implies that ϕ is continuous at t_0, and indeed we have a stronger result:

(1.1.2) *If $\phi:A\to Y$ has a derivative at t_0, then for each $\varepsilon>0$ there exists a neighbourhood U of t_0 in* **R** *such that for all $t\in U\cap A$*

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$$\|\phi(t)-\phi(t_0)\| \leq (\|\phi'(t_0)\|+\varepsilon)|t-t_0|.$$

This is an easy consequence of (1.1.1) (ii).

(1.1.3) (i) *If $\phi: A \to Y$ has a derivative at t_0, and α is a scalar, then $\alpha\phi$ has a derivative at t_0 equal to $\alpha\phi'(t_0)$.*
(ii) *If $\phi: A \to Y$, $\psi: B \to Y$ have derivatives at t_0, and t_0 is a non-isolated point of $A \cap B$, then $\phi+\psi$ has a derivative at t_0, equal to $\phi'(t_0)+\psi'(t_0)$.*

This too is immediate. We note that if t_0 is an interior point of A and B, then it is also an interior point of the domain $A \cap B$ of $\phi+\psi$.

This result implies in particular that if $A \subseteq \mathbf{R}$ and t_0 is a given non-isolated point of A, then the class of functions $\phi: A \to Y$ such that $\phi'(t_0)$ exists is a vector space under the operations of addition of functions and multiplication of functions by scalars. Moreover, the function $\phi \mapsto \phi'(t_0)$ is a linear functional on this space.

It should be noted that in general $(\phi+\psi)' \neq \phi'+\psi'$, since the domain of $\phi'+\psi'$ may be a proper subset of that of $(\phi+\psi)'$ (take, for example, $\phi(t) = -\psi(t) = |t|$).

(1.1.4) (Differentiation of a product and a quotient) *Let Y be a real normed space, and let $\phi: A \to \mathbf{R}$, $\psi: B \to Y$ be given functions, where $A, B \subseteq \mathbf{R}$. If t_0 is a non-isolated point of $A \cap B$, and $\phi'(t_0)$, $\psi'(t_0)$ exist, then the function $t \mapsto \phi(t)\psi(t)$ has a derivative at t_0 equal to $\phi(t_0)\psi'(t_0)+\phi'(t_0)\psi(t_0)$. If in addition $\phi(t_0) \neq 0$, then also the function $t \mapsto \psi(t)/\phi(t)$ has a derivative at t_0 equal to $(\phi(t_0)\psi'(t_0)-\phi'(t_0)\psi(t_0))/(\phi(t_0))^2$. The same result holds if Y is a complex normed space and ϕ maps A into $\mathbf{C}$.*

Here again the proofs are elementary and we omit them.

The next result is a generalization of the rule for differentiation of a product.

(1.1.5) *Let $Y_1, \ldots, Y_m$, Z be normed vector spaces, all real or all complex, and let $u: Y_1 \times \ldots \times Y_m \to Z$ be a continuous multilinear function. Let also $\phi_i: A_i \to Y_i$ $(i = 1, \ldots, m)$ be given functions, where each $A_i \subseteq \mathbf{R}$, let $A = \bigcap_{i=1}^m A_i$, and define $\phi: A \to Z$ by $\phi(t) = u(\phi_1(t), \ldots, \phi_m(t))$. If t_0 is a non-isolated point of A, and $\phi_i'(t_0)$ exists for each i, then $\phi'(t_0)$ exists and is equal to*

$$\sum_{i=1}^{m} u(\phi_1(t_0), \ldots, \phi_{i-1}(t_0), \phi_i'(t_0), \phi_{i+1}(t_0), \ldots, \phi_m(t_0)).$$

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Let $a_i = \phi_i(t_0), b_i = \phi_i(t) \quad (i = 1, \ldots, m;\ t \in A, t \neq t_0)$. Then

$$\phi(t) - \phi(t_0) = u(b_1, \ldots, b_m) - u(a_1, \ldots, a_m)$$
$$= \sum_{i=1}^{m} u(b_1, \ldots, b_{i-1}, b_i - a_i, a_{i+1}, \ldots, a_m).$$

Hence

$$\frac{\phi(t) - \phi(t_0)}{t - t_0} = \sum_{i=1}^{m} u\left(b_1, \ldots, b_{i-1}, \frac{b_i - a_i}{t - t_0}, a_{i+1}, \ldots, a_m\right),$$

and on making $t \to t_0$ in A we obtain the result.

The 'product rule' in (1.1.4) is the case where u is $(\alpha, y) \mapsto \alpha y$.

(1.1.6) (The chain rule) *Let $A, B \subseteq \mathbf{R}$, and suppose that $\phi : A \to \mathbf{R}$ has a derivative at t_0, that $\psi : B \to Y$ has a derivative at s_0, where $\phi(t_0) = s_0$, and that t_0 is a non-isolated point of the domain of $\psi \circ \phi$. Then $\psi \circ \phi$ has a derivative at t_0, equal to $\phi'(t_0)\psi'(s_0)$.*

(We recall that the domain $\mathscr{D}(\psi \circ \phi)$ of $\psi \circ \phi$ is the set of $t \in A$ for which $\phi(t) \in B$, i.e. the set $\phi^{-1}(B)$.)

By (1.1.1) (ii), we can write

$$\phi(t) = \phi(t_0) + (t - t_0)\lambda(t), \tag{2}$$
$$\psi(s) = \psi(s_0) + (s - s_0)\mu(s), \tag{3}$$

where the functions $\lambda : A \to \mathbf{R}$, $\mu : B \to Y$ are continuous at t_0, s_0, respectively, and $\lambda(t_0) = \phi'(t_0), \mu(s_0) = \psi'(s_0)$. Let $t \in \mathscr{D}(\psi \circ \phi)$, so that $\phi(t) \in B$. Then, from (3) with $s = \phi(t)$, $s_0 = \phi(t_0)$, and (2),

$$\psi(\phi(t)) = \psi(\phi(t_0)) + (t - t_0)\lambda(t)\mu(\phi(t)).$$

Since $t \mapsto \lambda(t)\mu(\phi(t))$ is continuous at t_0 and has there the value

$$\lambda(t_0)\mu(\phi(t_0)) = \phi'(t_0)\psi'(s_0),$$

the result follows from a further application of (1.1.1) (ii).

We note that if t_0 is an interior point of A and s_0 is an interior point of B, then t_0 is an interior point of $\mathscr{D}(\psi \circ \phi)$. For B is then a neighbourhood of s_0 in $\mathbf{R}$, and since $\phi(t_0) = s_0$ and ϕ is continuous at t_0, it follows that $\phi^{-1}(B)$ is a neighbourhood of t_0 in A. Further, A is a neighbourhood of t_0 in $\mathbf{R}$, whence $\phi^{-1}(B)$ is a neighbourhood of t_0 in $\mathbf{R}$, i.e. t_0 is an interior point of $\mathscr{D}(\psi \circ \phi)$.

(1.1.7) *Let ϕ be real-valued, strictly monotone, and continuous on an interval I, and let t_0 be a point of I at which ϕ has a derivative $\phi'(t_0) \neq 0$.*

Then the inverse function of ϕ *has a derivative at the point* $s_0 = \phi(t_0)$, *equal to* $1/\phi'(t_0)$.

Obviously ϕ is one-to-one, so that ϕ^{-1} is well-defined. Since, in addition, ϕ is continuous and therefore maps compact sets to compact sets, ϕ^{-1} is continuous. The function λ of (1.1.1) (ii) is continuous at t_0, and since $\lambda(t_0) \neq 0$, $1/\lambda$ is continuous at t_0. This is equivalent to the assertion.

Exercises 1.1

1 Let ϕ be a mapping of the interval $[a,b]$ into the normed space Y, and let $a < t < b$. Prove that
(i) if $\phi'(t)$ exists then $(\phi(v) - \phi(u))/(v-u)$ tends to $\phi'(t)$ as u, v tend to t in such a manner that $u < t < v$,
(ii) if ϕ is continuous at t and $(\phi(v) - \phi(u))/(v-u)$ tends to l as u, v tend to t in such a manner that $u < t < v$, then $\phi'(t)$ exists and is equal to l.

2 Let p be a continuous convex function on Y, and let ϕ be a function from a subset of $\mathbf{R}$ into Y possessing a right-hand derivative at t. Prove that $p \circ \phi$ has a right-hand derivative at t.
[Hint. Use the fact that if $y, z \in Y$ then $h \mapsto p(y + hz)$ is convex on $\mathbf{R}$.]

3 Let U, V be functions from a set $A \subseteq \mathbf{R}$ into the space $\mathscr{L}(Y, Y)$ of continuous linear maps of Y into itself. Prove that if U, V possess derivatives at a point $t_0 \in A$, then the function $t \mapsto U(t) \circ V(t)$ has a derivative at t_0 equal to $U'(t_0) \circ V(t_0) + U(t_0) \circ V'(t_0)$.

4 Let $A, B \subseteq \mathbf{R}$, and let $\phi : A \to \mathbf{R}$ and $\psi : B \to Y$ be functions such that ϕ has a right-hand derivative at t_0 and ψ has a derivative at an interior point s_0 of B, where $\phi(t_0) = s_0$. Prove that $\psi \circ \phi$ has a right-hand derivative at t_0 equal to $\phi'_+(t_0)\psi'(s_0)$.

1.2 Tangents to paths

It is familiar that the existence of the derivative of a real-valued function at a point t is equivalent to the existence of a tangent to the graph of the function at t. We prove here the analogous result for vector-valued functions, and we consider also the more general notion of a tangent to a path.

A *path* in a normed vector space Y is a continuous function ψ from a compact interval $[a,b]$ into Y. For each $t \in [a,b]$ the point $\psi(t)$ in Y is called *the point t of the path*, and the points $\psi(a)$, $\psi(b)$ are called the *endpoints* of the path. The path is said to be *simple* if ψ is one-to-one; in this case ψ^{-1} is also continuous (since $[a,b]$ is compact), so that ψ is a homeomorphism of $[a,b]$ into Y.

The term 'path' has an obvious geometrical connotation, and intuitively we think of the point $\psi(t)$ as tracing out a continuous path in Y as t runs from a to b. If the path is simple, it does not cross itself.